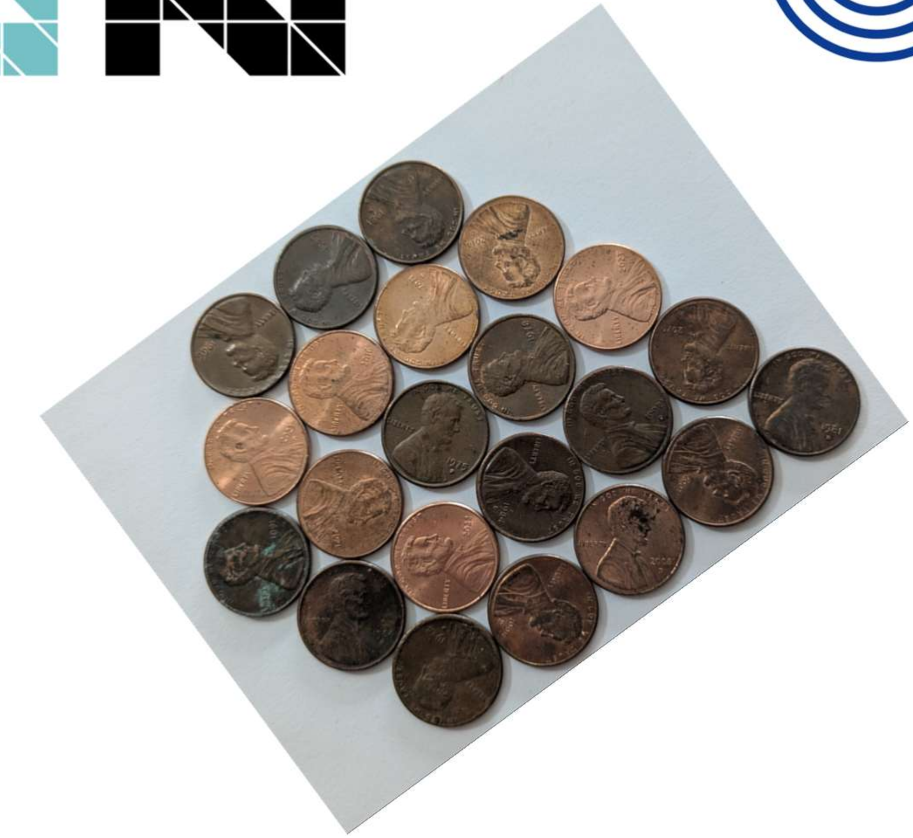
Calculating with Pennies and Marbles



Marcus Schaefer
DePaul University

Joint work with Anna Lubiw (University of Waterloo)

Cast

Pennies

Marbles

Matchsticks

NP …………………. *a complexity class*

$\exists\mathbb{R}$ ………………… *another complexity class*

Cast

Pennies

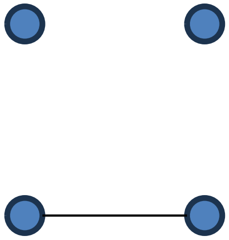
Marbles

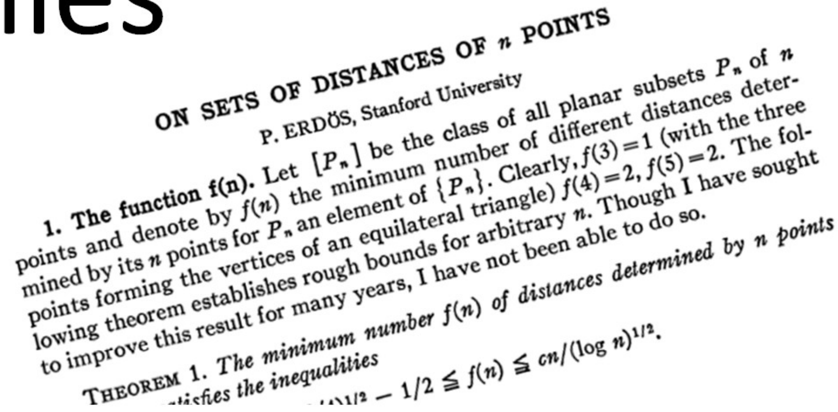
~~Matchsticks~~

NP …………………… *a complexity class*

$\exists\mathbb{R}$ ………………… *another complexity class*

Pennies





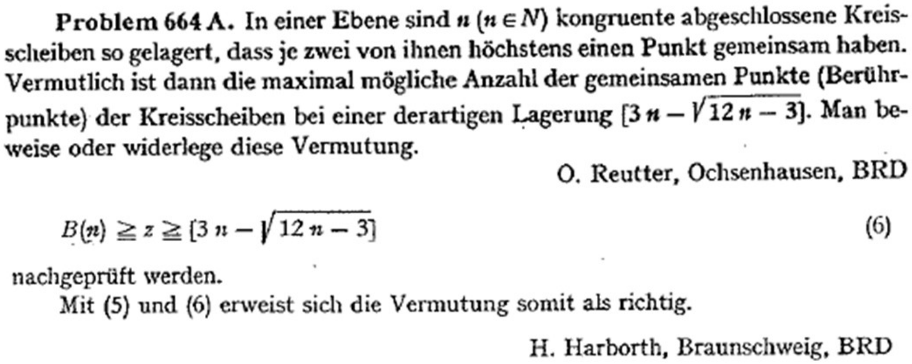
ON SETS OF DISTANCES OF n POINTS

P. ERDÖS, Stanford University

1. The function f(n). Let $[P_n]$ be the class of all planar subsets P_n of n points and denote by $f(n)$ the minimum number of different distances determined by its n points for P_n an element of $\{P_n\}$. Clearly, $f(3)=1$ (with the three points forming the vertices of an equilateral triangle) $f(4)=2$, $f(5)=2$. The following theorem establishes rough bounds for arbitrary n. Though I have sought to improve this result for many years, I have not been able to do so.

THEOREM 1. *The minimum number $f(n)$ of distances determined by n points tisfies the inequalities*

$$)^{1/2} - 1/2 \leqq f(n) \leqq cn/(\log n)^{1/2}.$$



Problem 664 A. In einer Ebene sind n ($n \in N$) kongruente abgeschlossene Kreisscheiben so gelagert, dass je zwei von ihnen höchstens einen Punkt gemeinsam haben. Vermutlich ist dann die maximal mögliche Anzahl der gemeinsamen Punkte (Berührpunkte) der Kreisscheiben bei einer derartigen Lagerung $[3n - \sqrt{12n - 3}]$. Man beweise oder widerlege diese Vermutung.

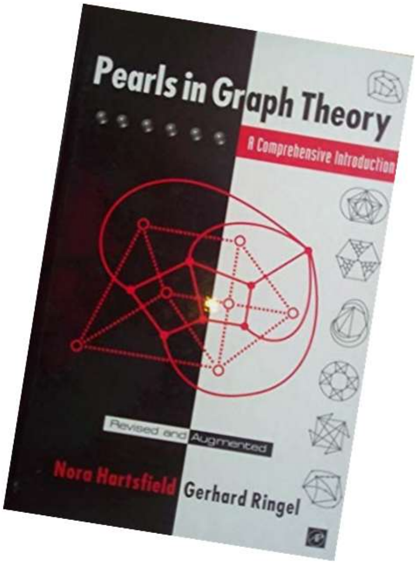
O. Reutter, Ochsenhausen, BRD

$$B(n) \geqq z \geqq [3n - \sqrt{12n - 3}] \qquad (6)$$

nachgeprüft werden.

Mit (5) und (6) erweist sich die Vermutung somit als richtig.

H. Harborth, Braunschweig, BRD



coin graphs = planar graphs
(Koebe's theorem)

Penny Graph Recognition

$K_4 - e \longrightarrow$

Theorem (Breu, Kirkpatrick, 1996/1998)

Recognizing penny graphs is NP-hard.

Theorem (Bowen, Durocher, Löffler, Rounds, Schulz, and Tóth, 2015)

Recognizing penny graphs is NP-hard if the graph is a tree.

Penny Graph Recognition

$K_4 - e$ →

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Why NP-hard ?

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Penny Graph Recognition

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Why NP-hard ?

decidable?

***54·43.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51{\cdot}231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

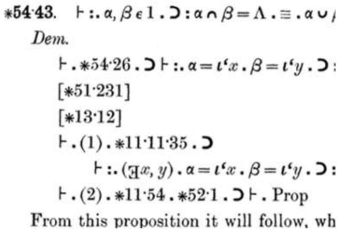
$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash .$ Prop

Definability



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$[*51{\cdot}231]$

$[*13{\cdot}12]$

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$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash .$ Prop

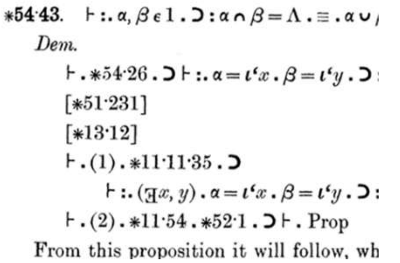
From this proposition it will follow, wh

$K_4 - e \longrightarrow$

$$\begin{aligned}
&(\exists\, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)\\
&\quad (x_1 - x_2)^2 + (y_1 - y_2)^2 = 1 \wedge\\
&\quad (x_1 - x_3)^2 + (y_1 - y_3)^2 = 1 \wedge\\
&\quad (x_2 - x_3)^2 + (y_2 - y_3)^2 = 1 \wedge\\
&\quad (x_2 - x_4)^2 + (y_2 - y_4)^2 = 1 \wedge\\
&\quad (x_3 - x_4)^2 + (y_3 - y_4)^2 = 1 \wedge\\
&\quad (x_1 - x_4)^2 + (y_1 - y_4)^2 > 1
\end{aligned}$$

✓

Existential Theory of the Reals



$*54{\cdot}43.\quad \vdash :. \alpha, \beta \in 1 \,.\, \supset : \alpha \cap \beta = \Lambda \,.\, \equiv \,.\, \alpha \cup$

Dem.

$\vdash . *54{\cdot}26 \,.\, \supset \vdash :. \alpha = \iota`x \,.\, \beta = \iota`y \,.\, \supset$

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$\vdash :. (\exists x, y) \,.\, \alpha = \iota`x \,.\, \beta = \iota`y \,.\, \supset$

$\vdash . (2) . *11{\cdot}54 \,.\, *52{\cdot}1 \,.\, \supset \vdash .$ Prop

From this proposition it will follow, wh

$$(\exists x)(\exists y)(\exists z)[(2xy - x^2 = z) \wedge (5xy = z^4 \vee 5xy = -z^5)]$$

Signature: +,-,·,

0, 1,

=, <

Boolean operators

ETR = set of all formulas in this signature true over the reals

Compare:

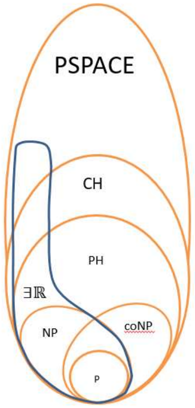
Existential Theory of GF[2]

Existential Theory of the Natural Numbers

Existential Theory of the Rationals

Existential Theory of the Complex Numbers

Existential Theory of the Reals



$(\exists x)(\exists y)(\exists z)[(2xy - x^2 = z$

S ,-,*

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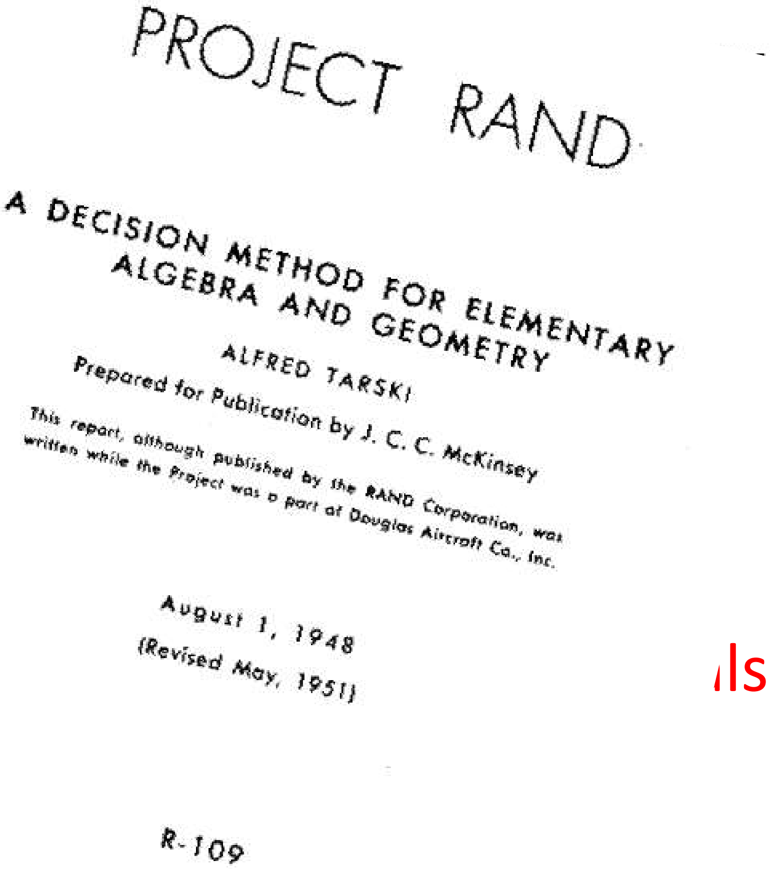
Bo



PROJECT RAND

A DECISION METHOD FOR ELEMENTARY ALGEBRA AND GEOMETRY

ALFRED TARSKI

Prepared for Publication by J. C. C. McKinsey

This report, although published by the RAND Corporation, was written while the Project was a part of Douglas Aircraft Co., Inc.

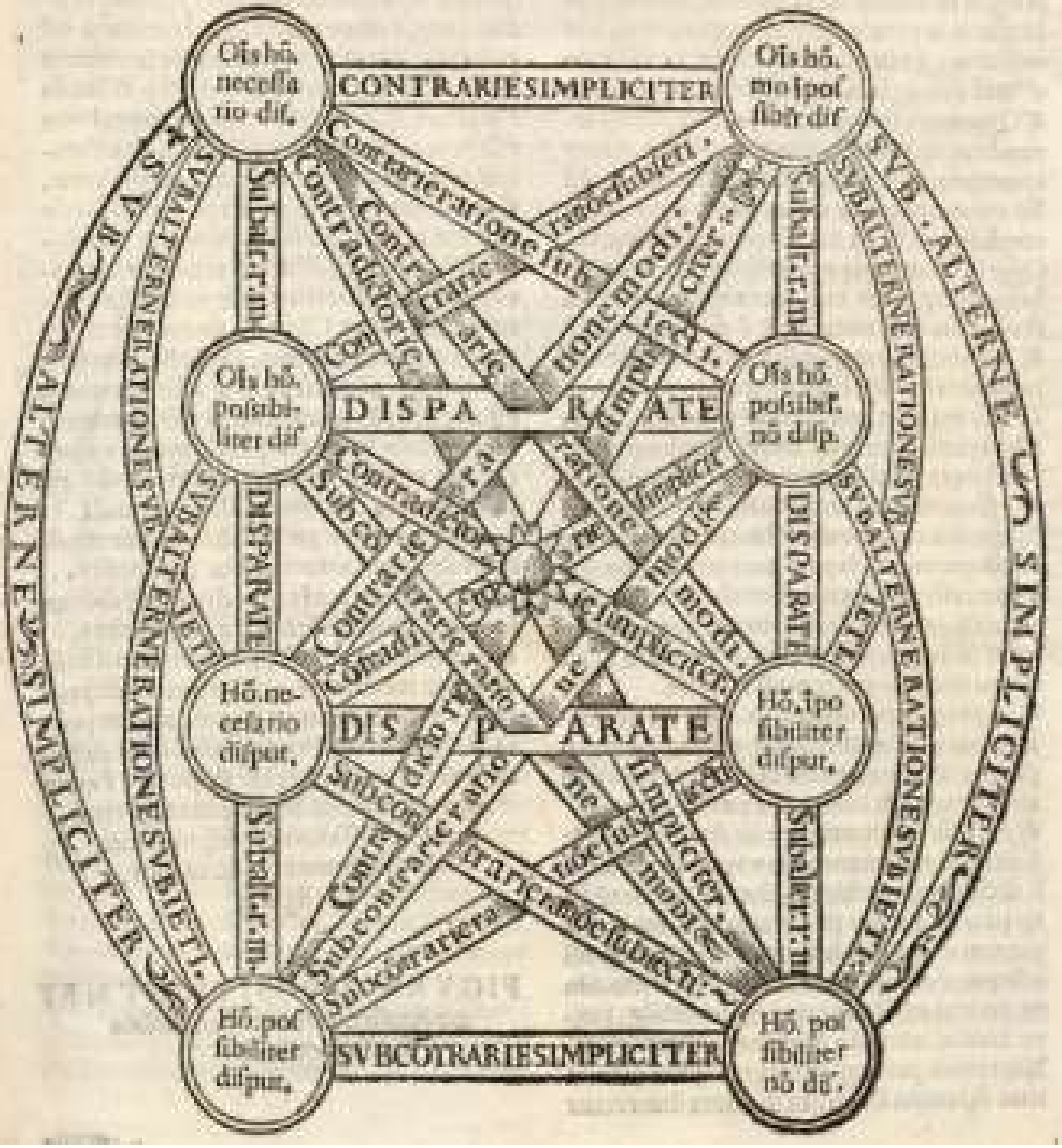
August 1, 1948
(Revised May, 1951)

R-109

ETR = set of all formulas ir ıls

Compare:

- Existential Theory of GF[2]
- Existential Theory of the Natural Numbers
- Existential Theory of the Rationals
- Existential Theory of the Complex Numbers


Oīs hō. neceſſario dīſ.
CONTRARIE SIMPLICITER
Oīs hō. mo īpoſſibiľr dīſ
Subalter.r.m.
Subalter.r.m.
Contrarie ratione ſubiecti.
Contradictorie
Contrarie ratione modi.
SVBALTERNE RATIONE SVBIECTI.
ALTERNE SIMPLICITER
SVB ALTERNE SIMPLICITER
Oīs hō. poſſibiliter dīſ
DISPA RATE
Oīs hō. poſſibiľr. nō diſp.
DISPARATE
DISPARATE
Subcontrarie
Contradictorie
Hō. neceſſario diſput.
DIS P ARATE
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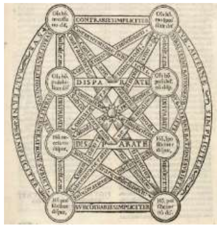
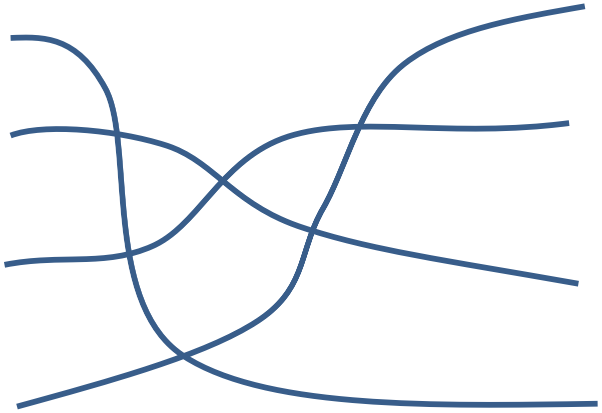
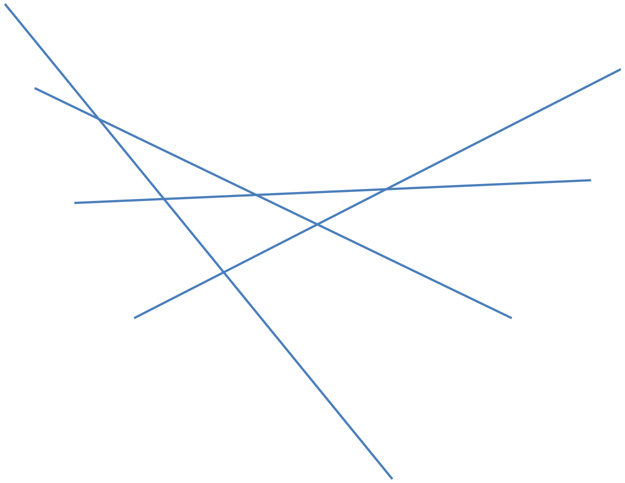
ſimpliciter.
Hō. poſſibiliter diſput.
SVBCŌTRARIE SIMPLICITER
Hō. poſſibiliter nō dīſ.

STRETCHABILITY

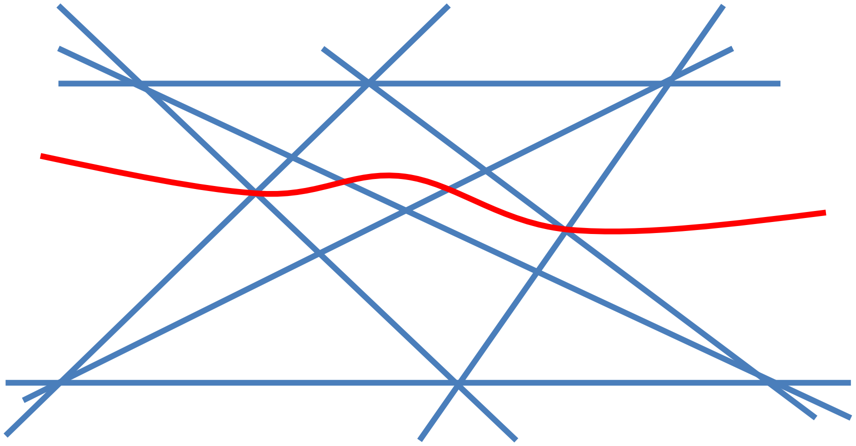




Pseudoline arrangement

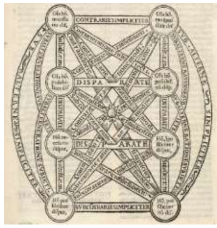


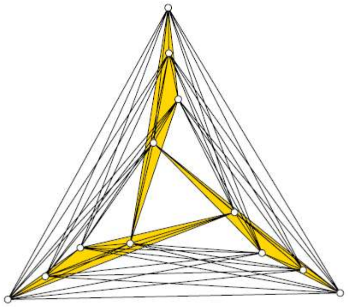
Equivalent line arrangement

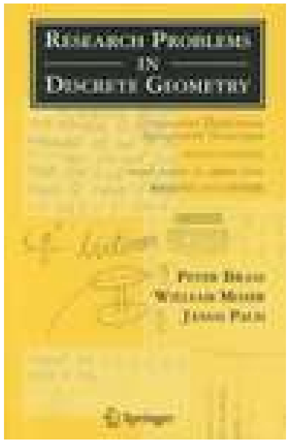


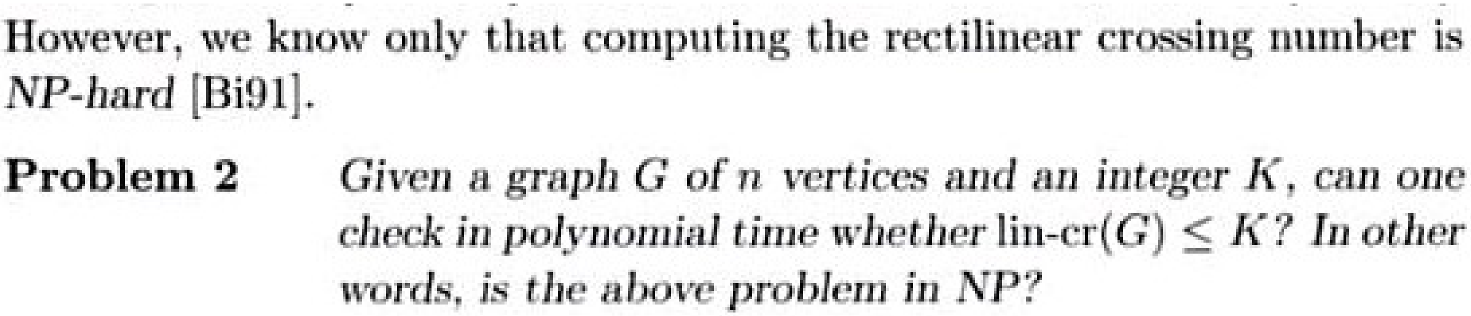
Not stretchable (Pappus’ Configuration)

Rectilinear Crossing Number





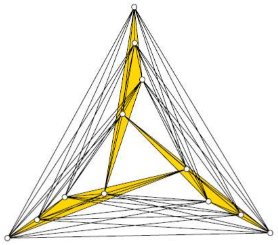




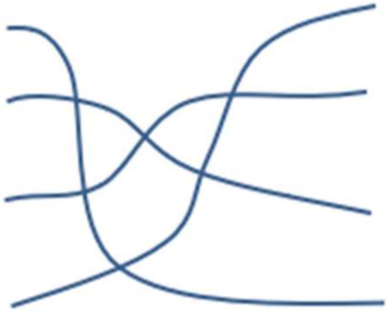
However, we know only that computing the rectilinear crossing number is *NP-hard* [Bi91].

Problem 2 *Given a graph G of n vertices and an integer K, can one check in polynomial time whether* $\text{lin-cr}(G) \leq K$*? In other words, is the above problem in NP?*

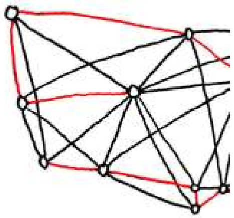
Power of ETR



Rectilinear Crossing Number



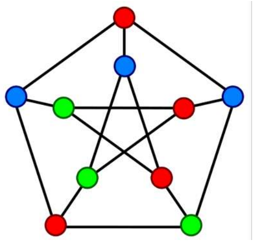
Pseudoline Stretchability



Euclidean TSP

$\exists x_1, x_2, \cdots \varphi(x_1, x_2, \ldots)$

SATISFIABILITY

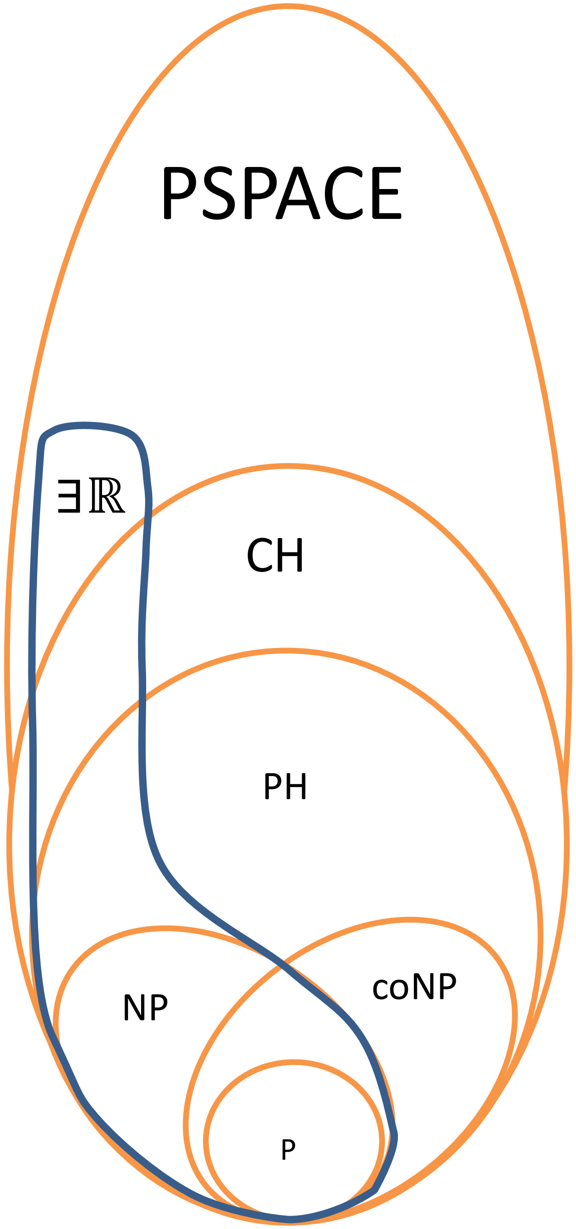


3-Colorability

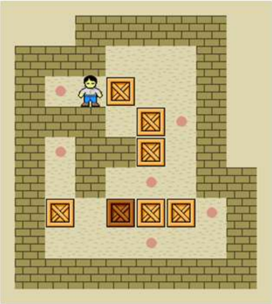
$\exists c_1, \ldots, c_{10}, x_1, y_1, \ldots x_{10}, y_{10},$
$c_1 = x_1 + y_1, x_1^2 = x_1, y_1^2 = y_1, \ldots,$
$c_{10} = x_{10} + y_{10}, x_{10}^2 = x_{10}, y_{10}^2 = y_{10},$
$c_1 \neq c_2, c_1 \neq c_5, c_1 \neq c_6, \ldots, c_8 \neq c_{10}$

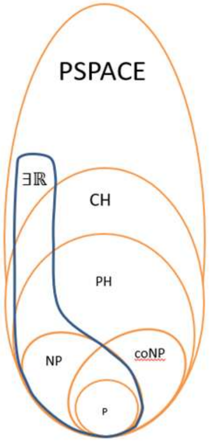
$\sqrt{3} + \sqrt{5} \overset{?}{<} \sqrt{7}$

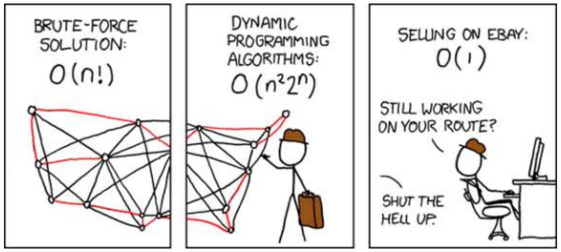
Sum of Square Roots

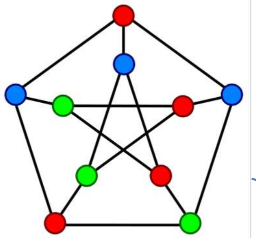

PSPACE
∃ℝ
CH
PH
NP
coNP
P

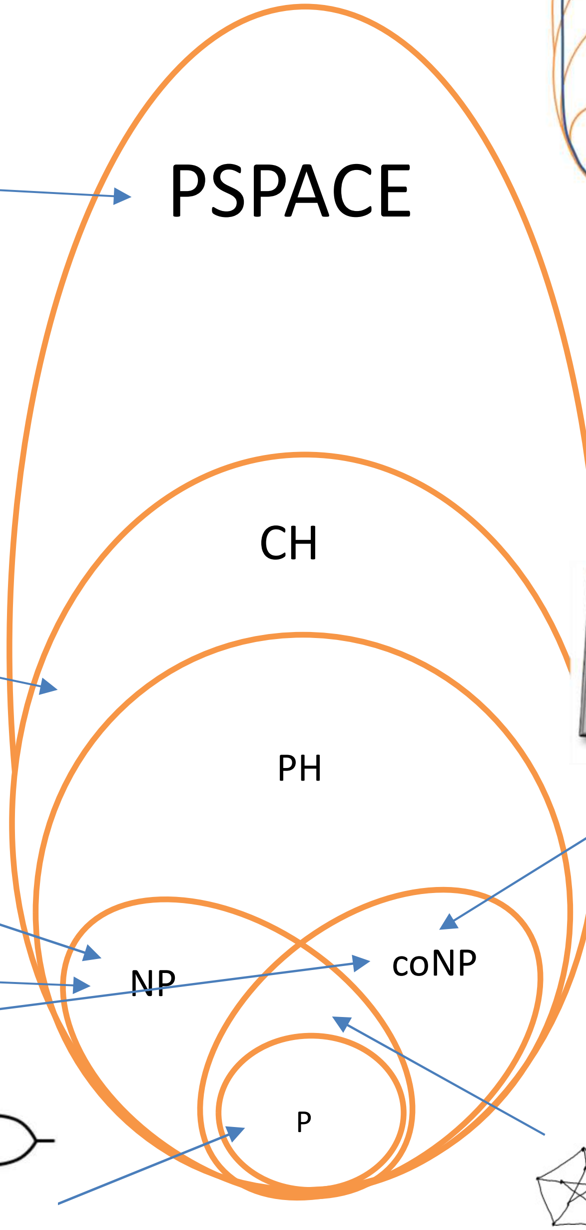
How Hard is This?









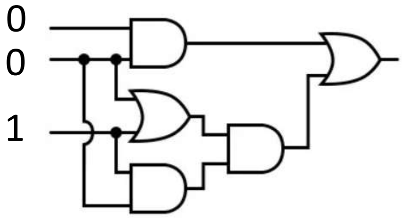


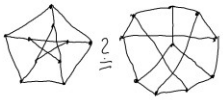
$\exists x_1, x_2, \cdots \varphi(x_1, x_2, \dots)$

SAT

TAUT

$\forall x_1, x_2, \cdots \varphi(x_1, x_2, \dots)$

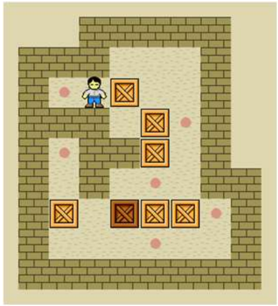


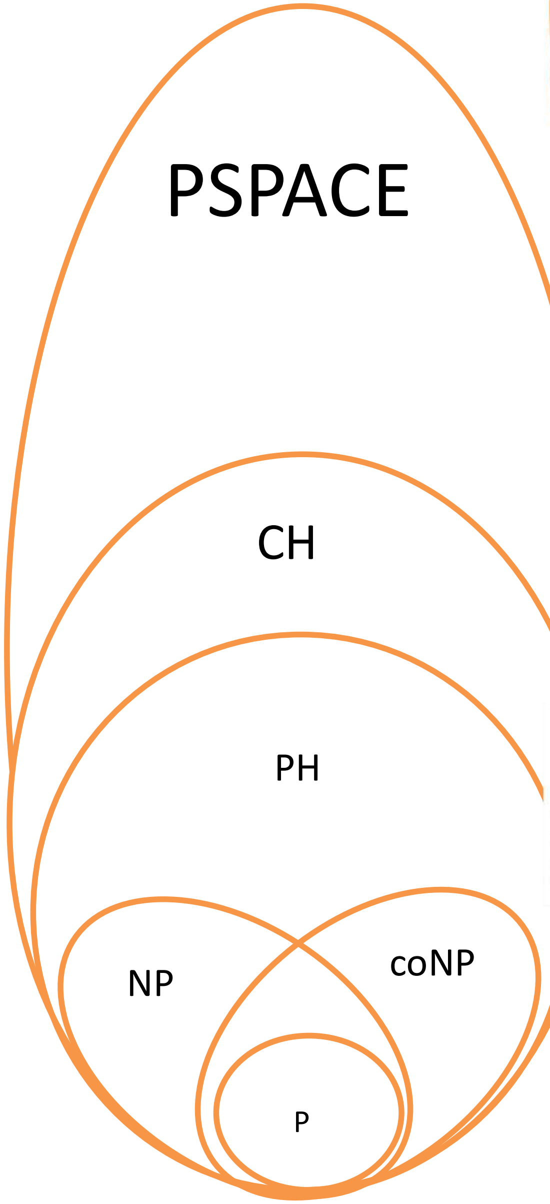


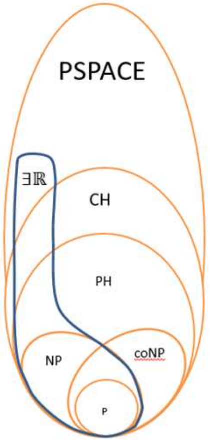
15 = ?*?

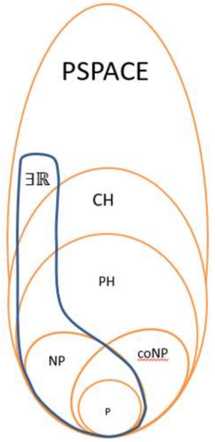
How Hard is This?

- measuring computational hardness directly is often difficult
- instead: use relative measure (reductions)
- need a notion of computationally easy: P (polynomial-time)









$\exists\mathbb{R}$

$$\exists\mathbb{R} \coloneqq \{L \subseteq \{0,1\}^* : L \leq \mathrm{ETR}\}$$

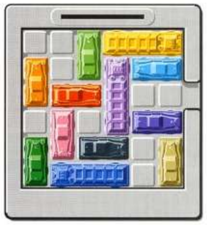
$$\mathrm{NP} \subseteq \exists\mathbb{R} \subseteq \mathrm{PSPACE}$$

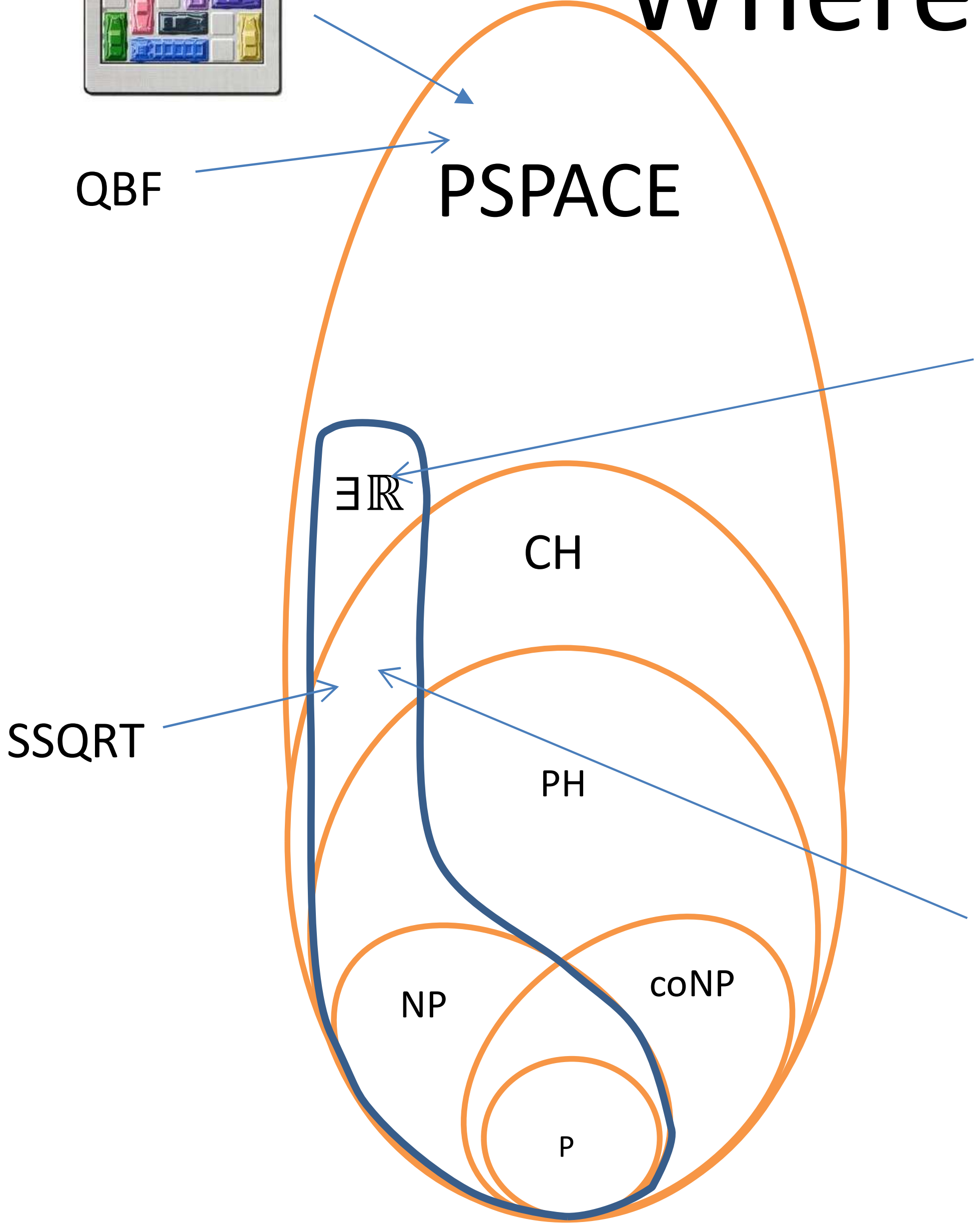
Canny, 1988

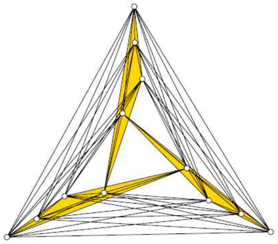
L is $\exists\mathbb{R}$-*hard*: for all $L' \in \exists\mathbb{R}$: $L' \leq L$

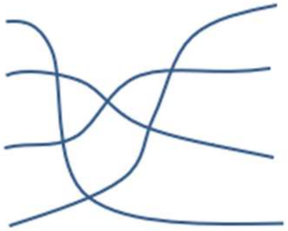
L is $\exists\mathbb{R}$-*complete*: $\exists\mathbb{R}$-hard and in $\exists\mathbb{R}$

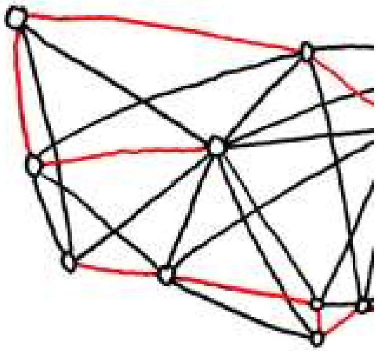
Where is ∃ℝ ?

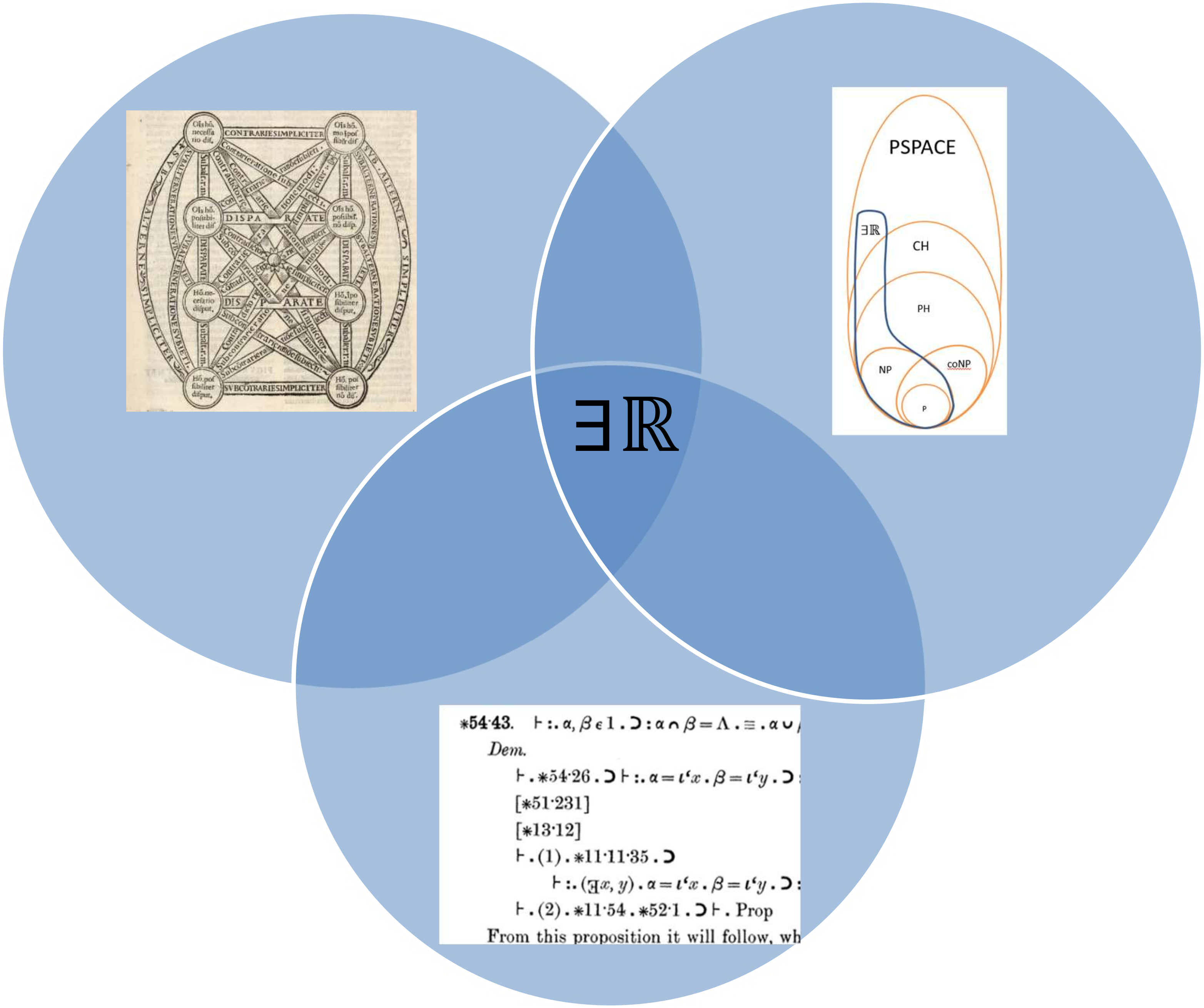












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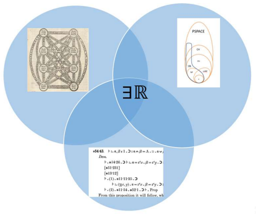
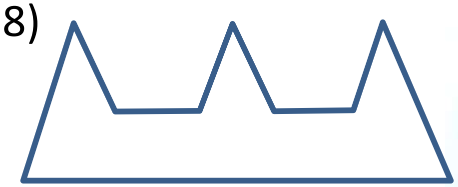
$\vdash .\, (2) \,.\, *11{\cdot}54 \,.\, *52{\cdot}1 \,.\, \supset \vdash .$ Prop

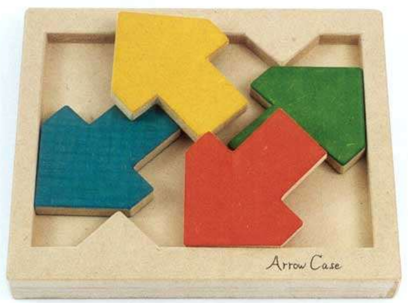
From this proposition it will follow, wh

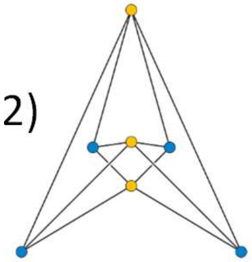
Some $\exists\mathbb{R}$-completeness Result



- Geometric Realizability of complete graph (Kyncl, 2007)
- Simultaneous Geometric Embeddability (Cardinal, Kusters 2014)
- Inscribed polytopes and Delaunay triangulations (Adiprasito, Padrol, Theran, 2015)
- Intersection graphs of
  - segments (Kratochvil, Matousek, 1994)
  - unit segments (Hoffmann, Miltzow, Weber, and Wulf, 2024)
  - lines in $\mathbb{R}^3$ (Cardinal 2024)
  - ellipses, convex sets (S 2008)
  - unit disks, disks (McDiarmid, Müller, 2010)
  - dot-product graphs (Kang, Müller, 2012)
  - rays and grounded segments (Cardinal, Felsner, Miltzow, Tompkins, Vogtenhuber, Birgit, 2018)
  - pseudocircles (Felsner, Scheucher, 2020)
- Topological Inference
  - with convexity (Davis, Gotts, Cohn, 1999)
- Art Gallery Problem (Abrahamsen, Adamaszek, Miltzow, 2017)
- Planar Slope Number (Hoffmann, 2016)
- Allowable Sequence realizabilty (Hoffmann, 2017)
- Visibility Graphs (Cardinal, Hoffmann, 2017)
- Packing Problem (Abrahamsen, Miltzow, Seiferth, 2020, Abrahamsen 2024)
- Spanners w/ ratio 1 (Aichholzer, Borrazzo, Bose, Cardinal, Frati, Morin, Vogtenhuber, 2022)
- Training Neural Networks (Abrahamsen, Kleist, Miltzow, 2021)
- RAC-drawability (S 23)
- Star-shapedness (Štefankovič, S 25)





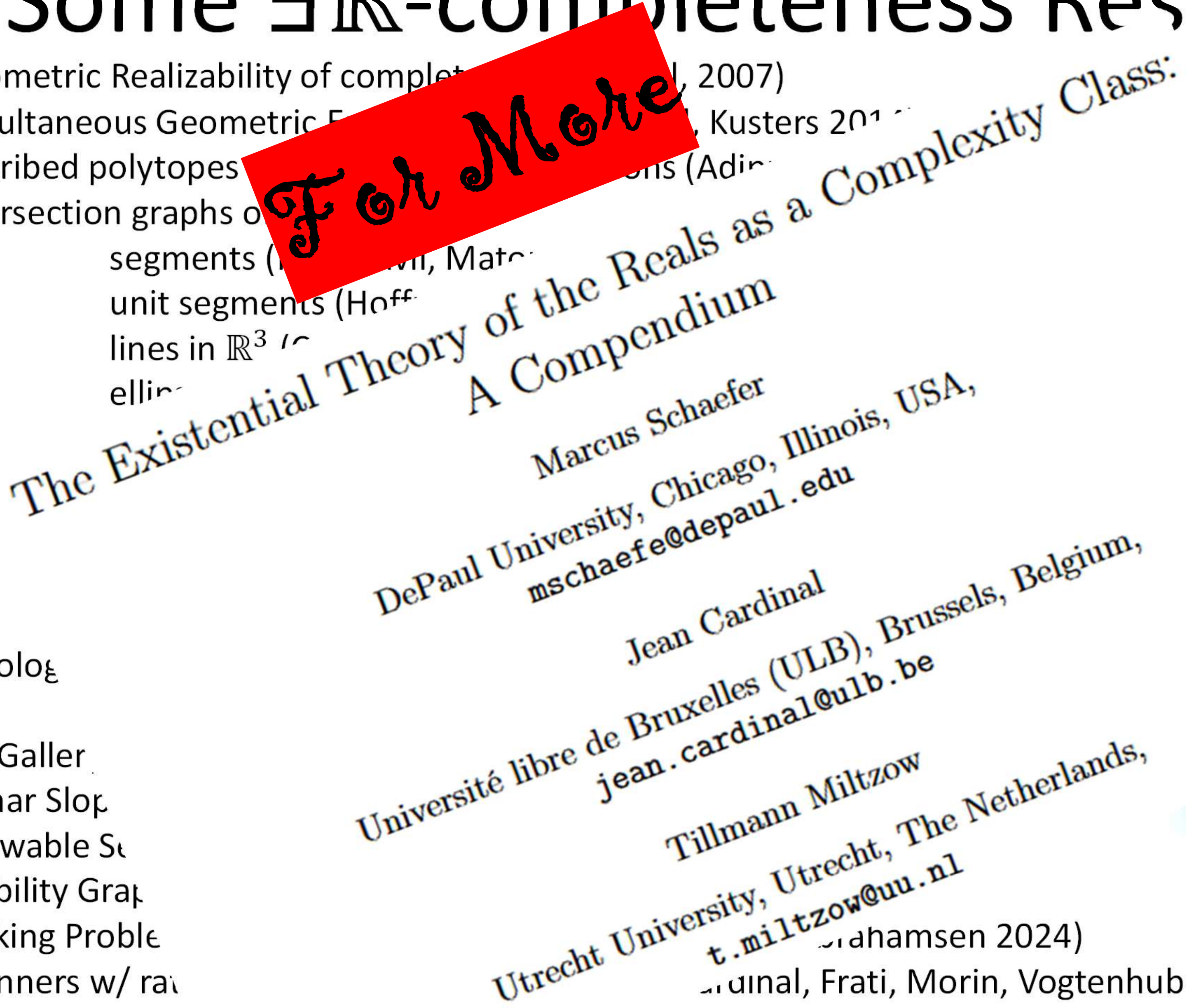


Some ∃ℝ-completeness Result

Geometric Realizability of comple… l, 2007)
Simultaneous Geometric E… , Kusters 201…
Inscribed polytopes … ons (Adir…
Intersection graphs o…
  segments (… vi, Mat…
  unit segments (Hoff…
  lines in $\mathbb{R}^3$ (…
  ellip…

Topolo…

Art Galler…
Planar Slop…
Allowable Se…
Visibility Gra…
Packing Proble… …ahamsen 2024)
Spanners w/ ra… …rdinal, Frati, Morin, Vogtenhuber, 2022)
Training Neural … …eist, Miltzow, 2021)
RAC-drawability (…
Star-shapedness (Š… …ic, S 25)



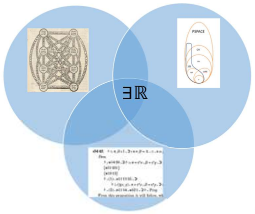
For More

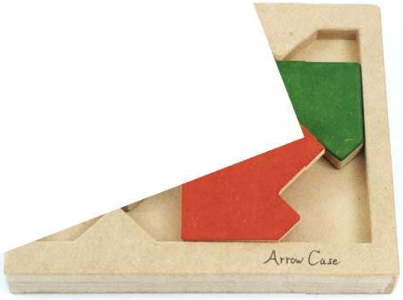
The Existential Theory of the Reals as a Complexity Class:
A Compendium

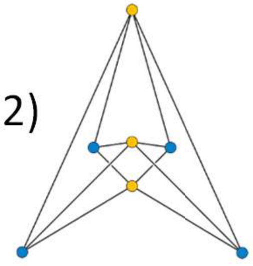
Marcus Schaefer
DePaul University, Chicago, Illinois, USA,
mschaefe@depaul.edu

Jean Cardinal
Université libre de Bruxelles (ULB), Brussels, Belgium,
jean.cardinal@ulb.be

Tillmann Miltzow
Utrecht University, Utrecht, The Netherlands,
t.miltzow@uu.nl







Calculating with Pennies

Theorem (Lubiw, Schaefer, 2025)

Recognizing penny graphs is $\exists\mathbb{R}$-complete even if a combinatorial embedding is given.

Proving $\exists\mathbb{R}$-hardness

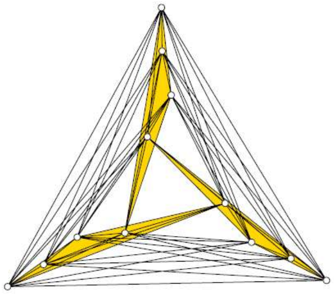
Algebraic Encoding

encode

- real numbers,
- addition, multiplication
- (in)equalities

Stretchability Encoding

encode pseudoline incidences



Proving ∃ℝ-hardness

Algebraic Encoding

encode

- real numbers,
- addition, multiplication
- (in)equalities

Stretchability Encoding

encode pseudoline incidences

ETRINV

Is there $x \in \left(\frac{1}{2}, 2\right)^n$ satisfying equations of types

$$x_i = x_j + x_k$$

$$x_i x_j = 1$$

∃ℝ-complete by Abrahamsen, Adamaszek, and Miltzow, 2018

$\exists\mathbb{R}$-hardness of Penny Graphs

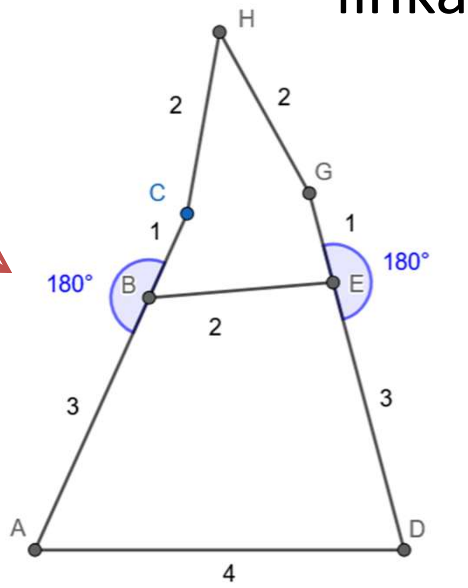
ETRINV

Is there $x \in \left(\frac{1}{2}, 2\right)^n$ satisfying equations of types

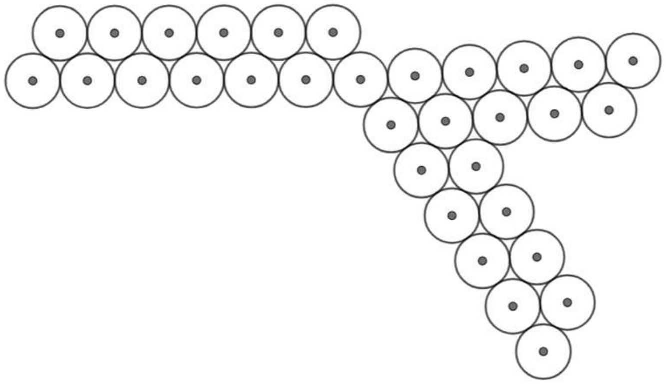
$$x_i = x_j + x_k$$
$$x_i x_j = 1$$

Step 1

angle-constrained linkage



Step 2

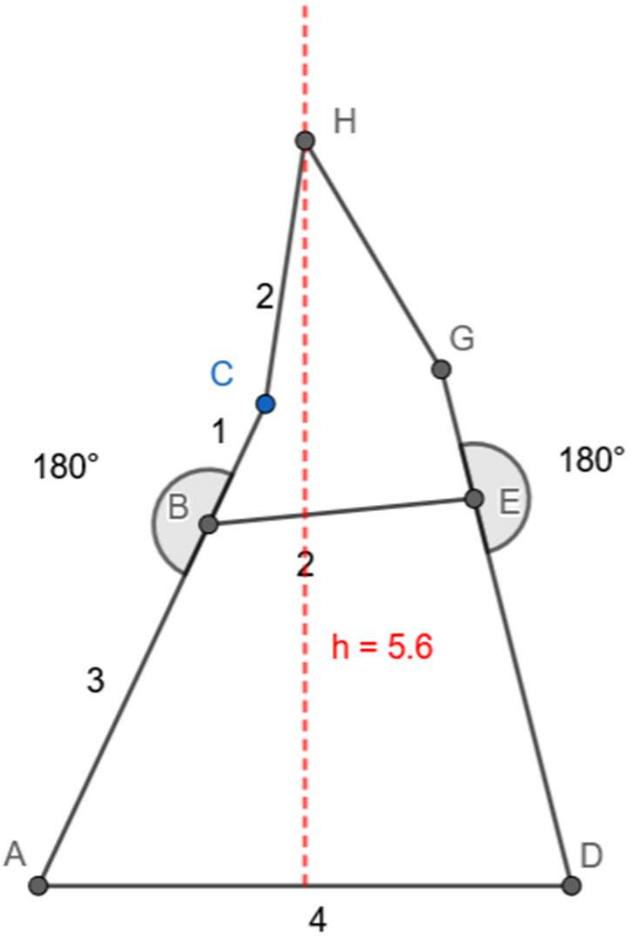


penny graph

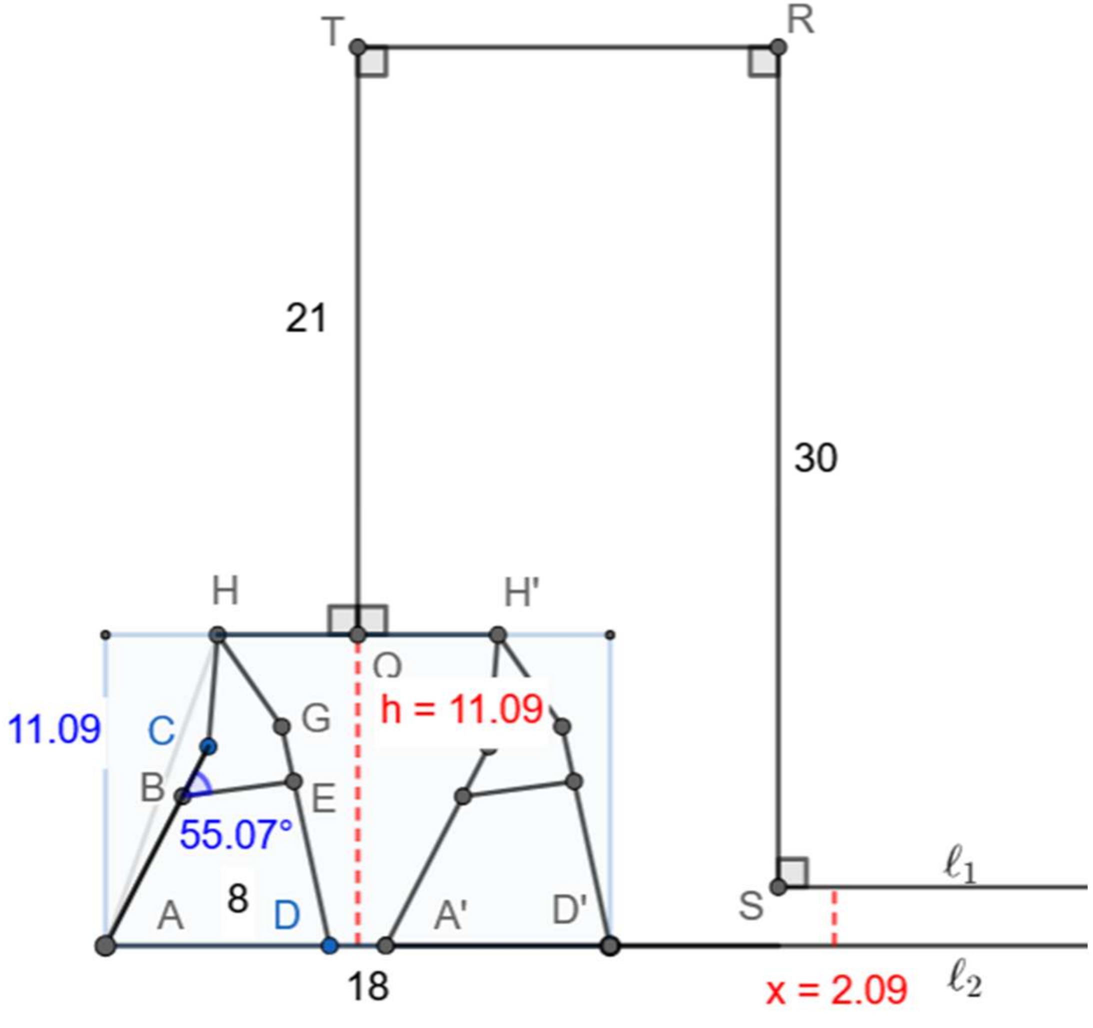
STEP 1

ETRINV → LINKAGES

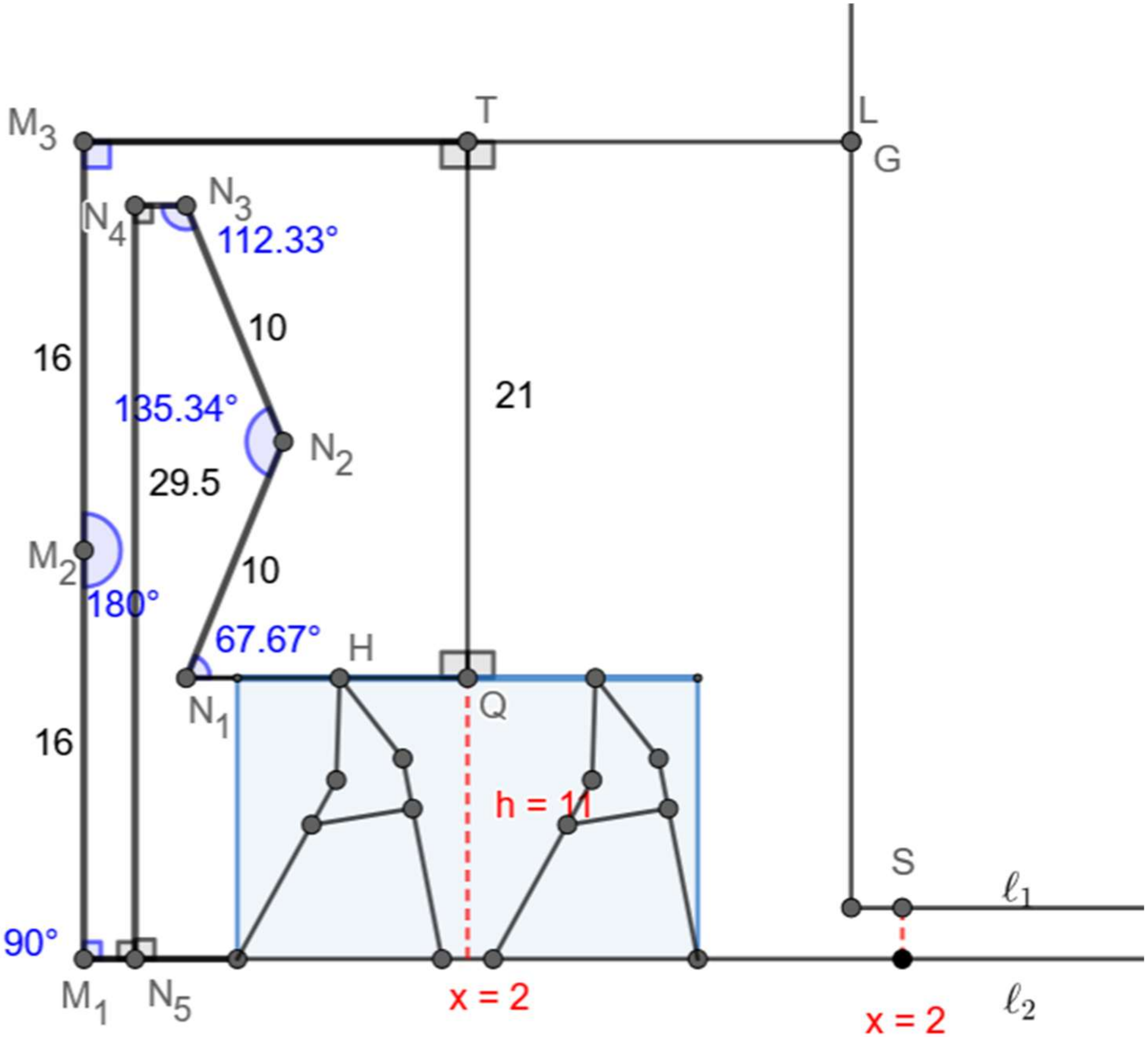
Hart’s A-Frame



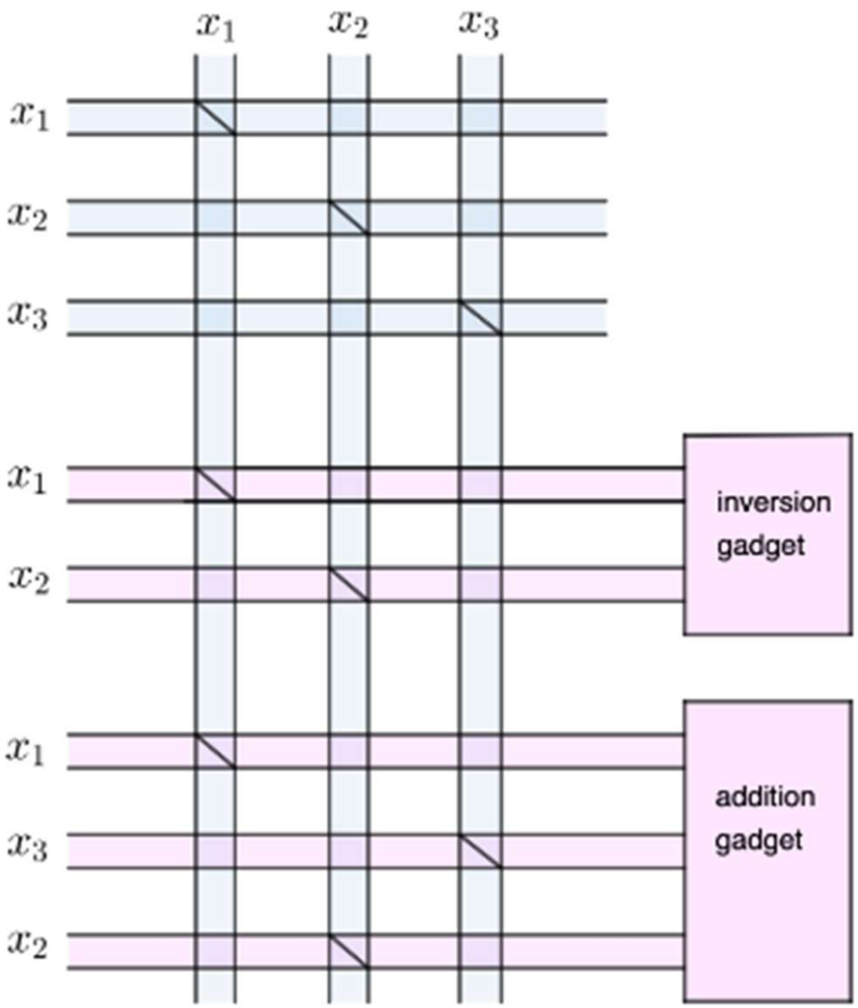
The Flex Gadget



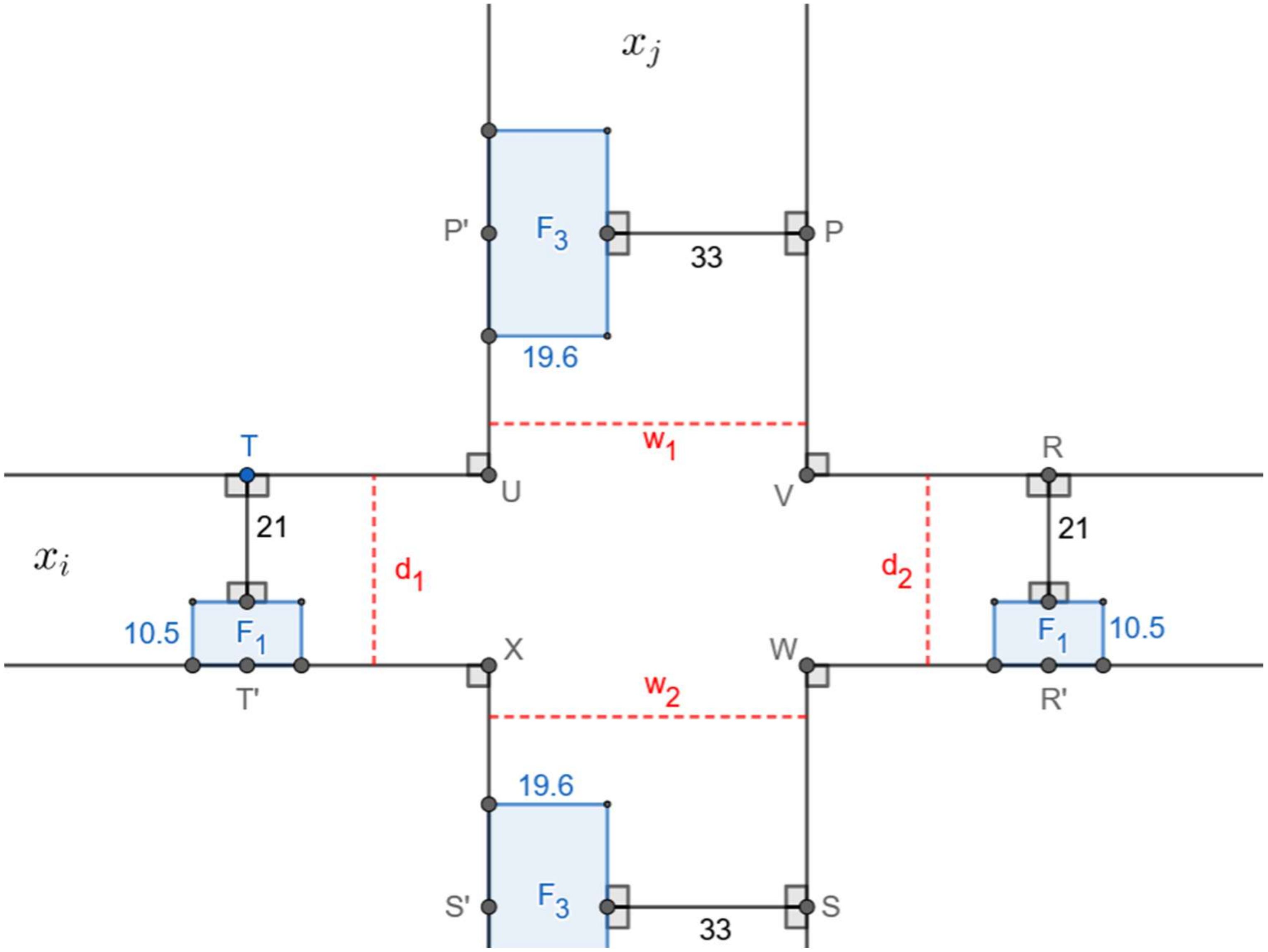
The Variable Gadget



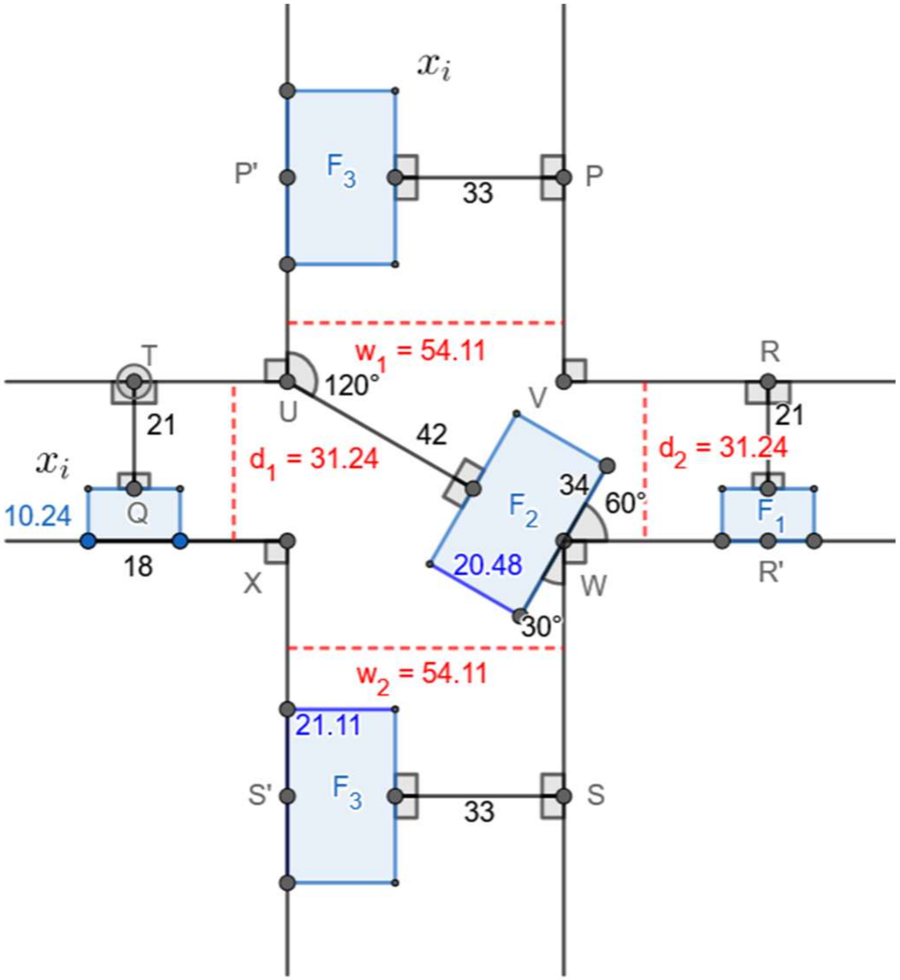
Construction Overview



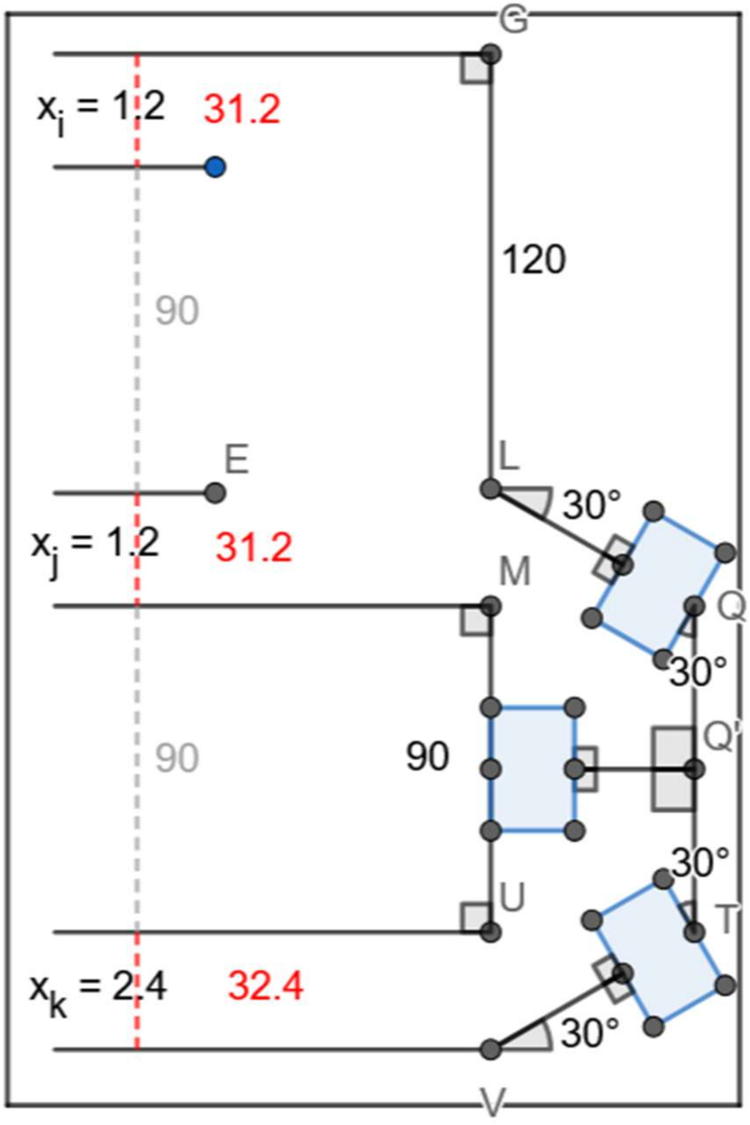
Cross-Over Gadget I



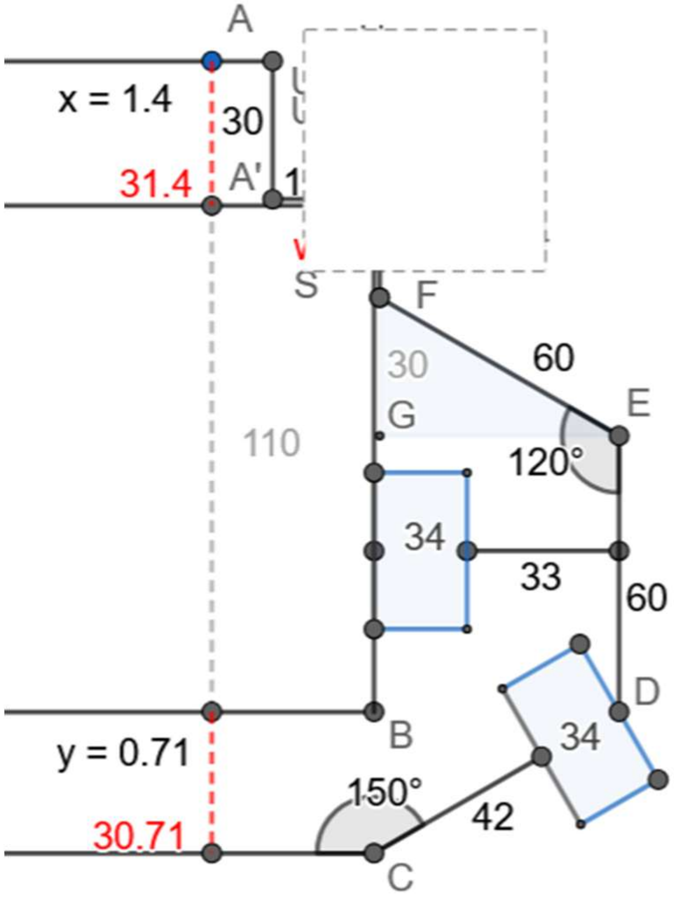
Cross-Over Gadget II



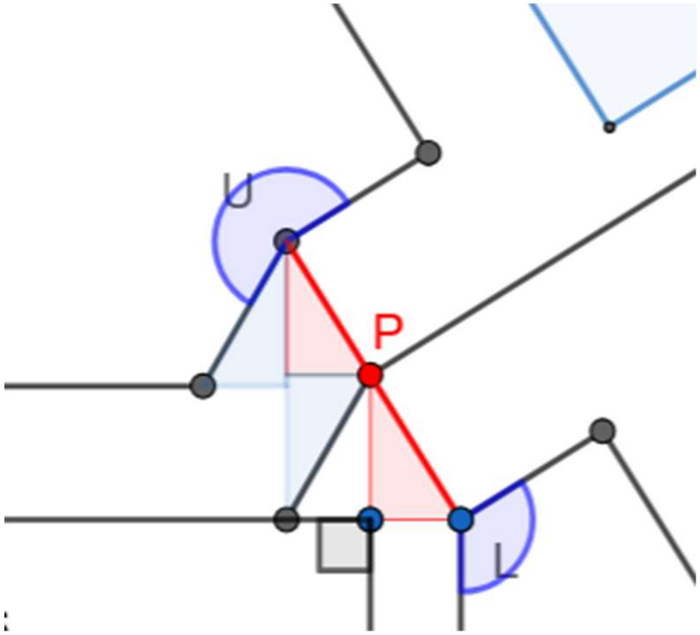
Addition Gadget



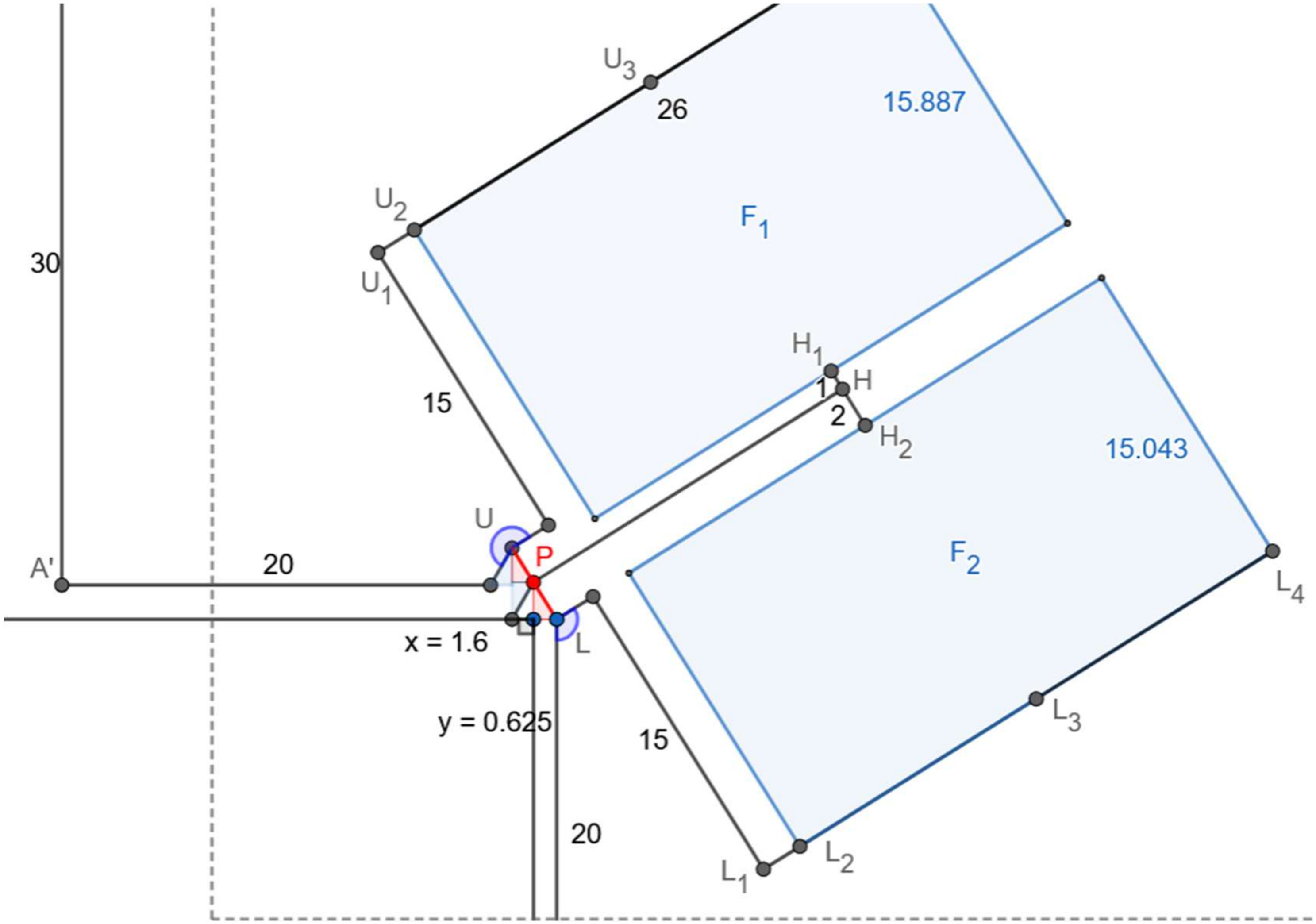
Inversion Gadget I



Inversion Gadget II



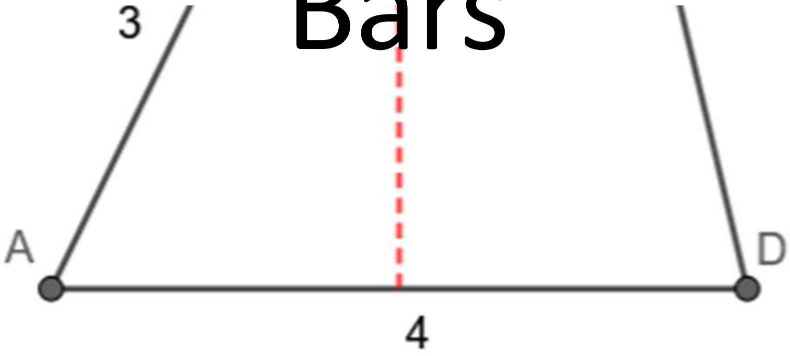
Inversion Gadget III

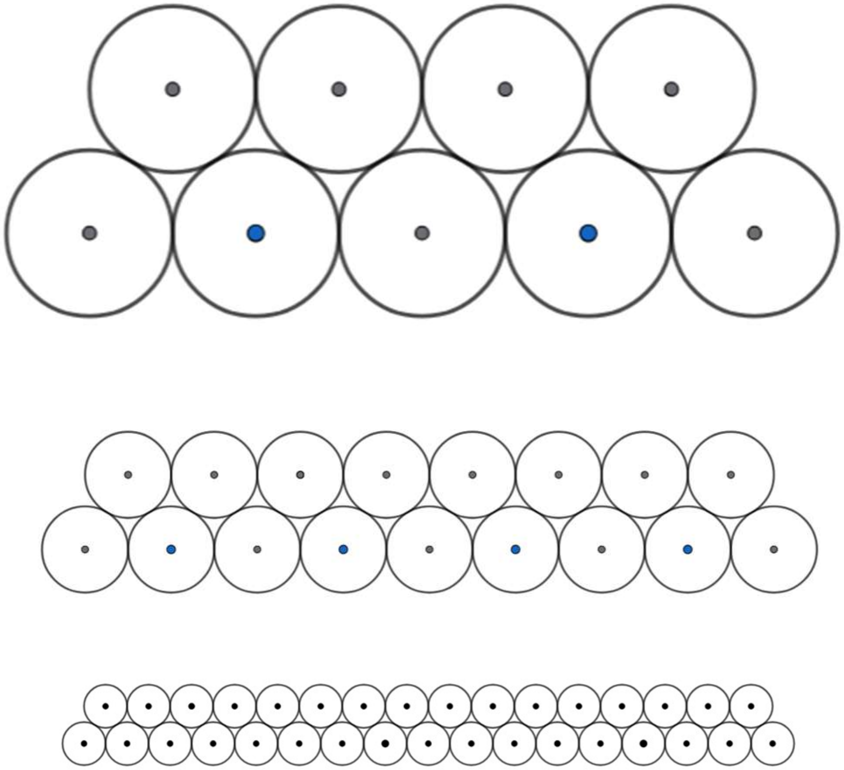


STEP 2

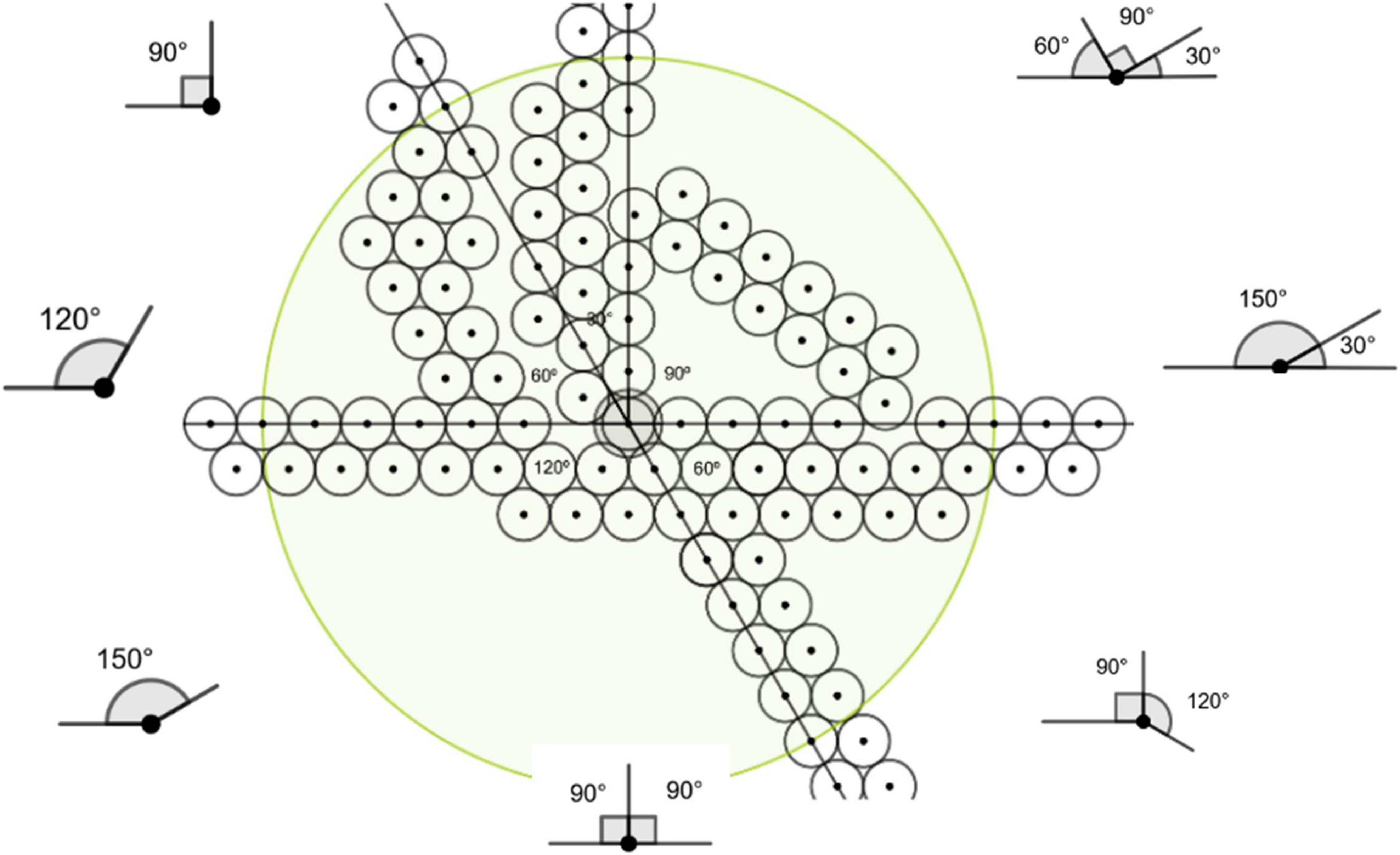
LINKAGES → PENNY GRAPHS

Bars

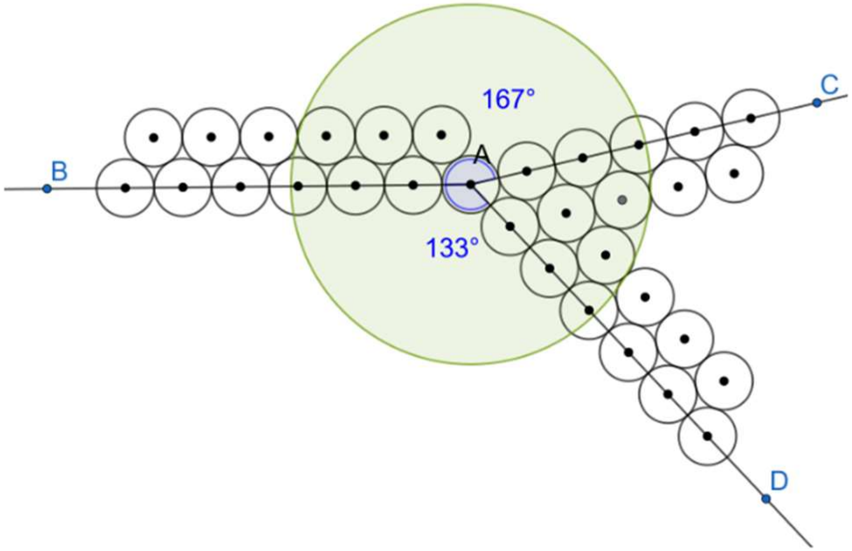


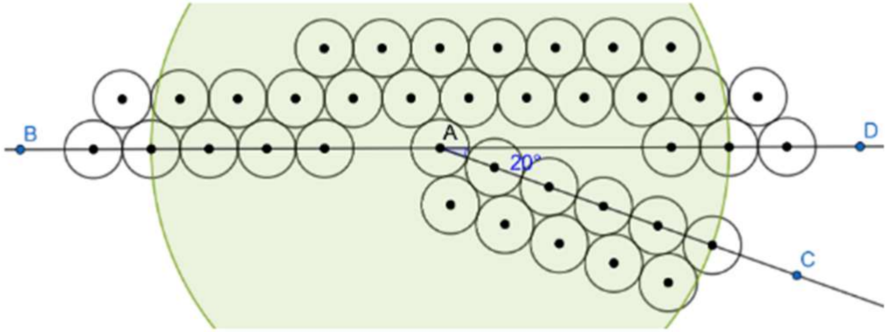


Fixed Angle Vertices

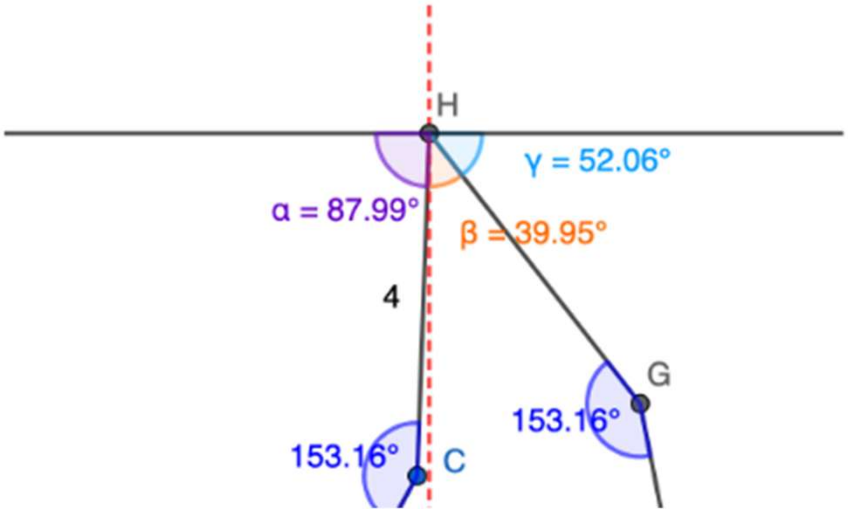


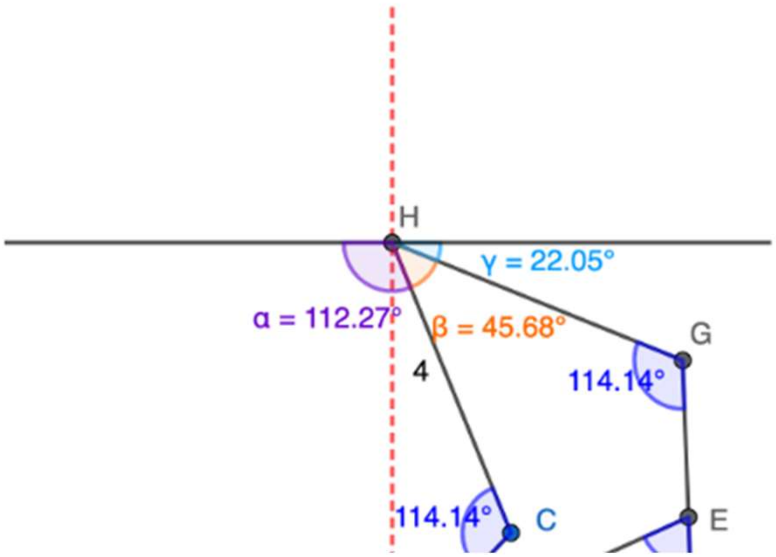
Flexible Angle Vertices I



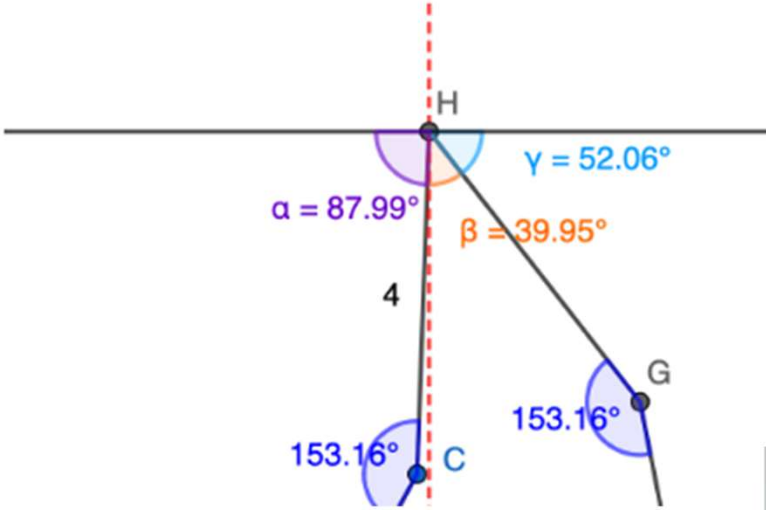


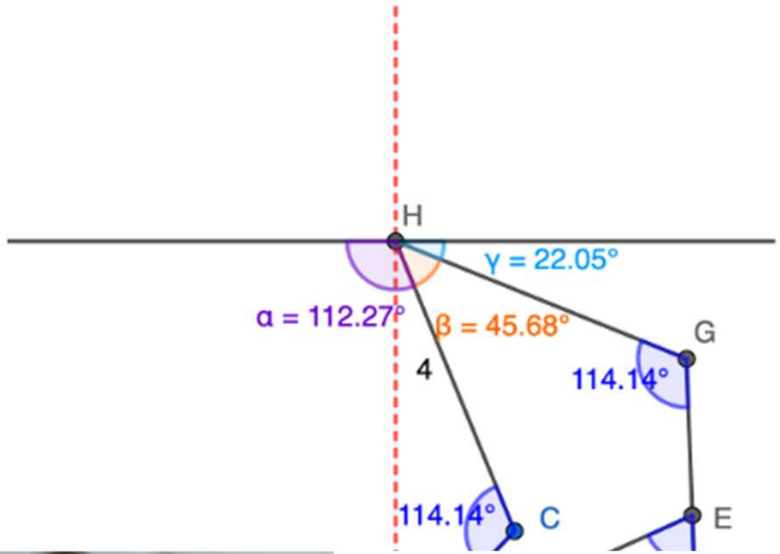
Flexible Angle Vertices II

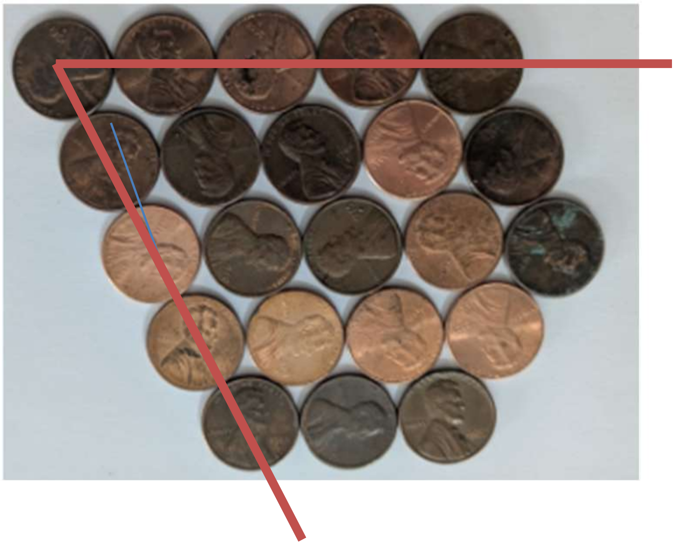




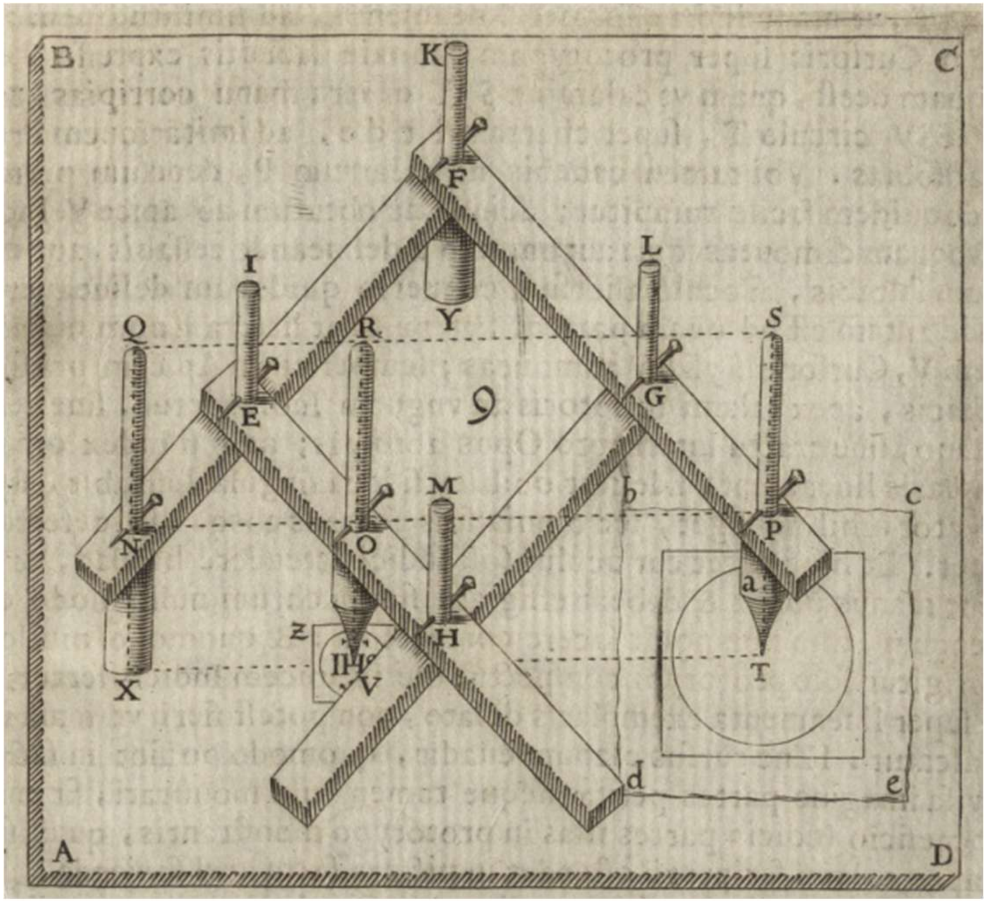
Flexible Angle Vertices II





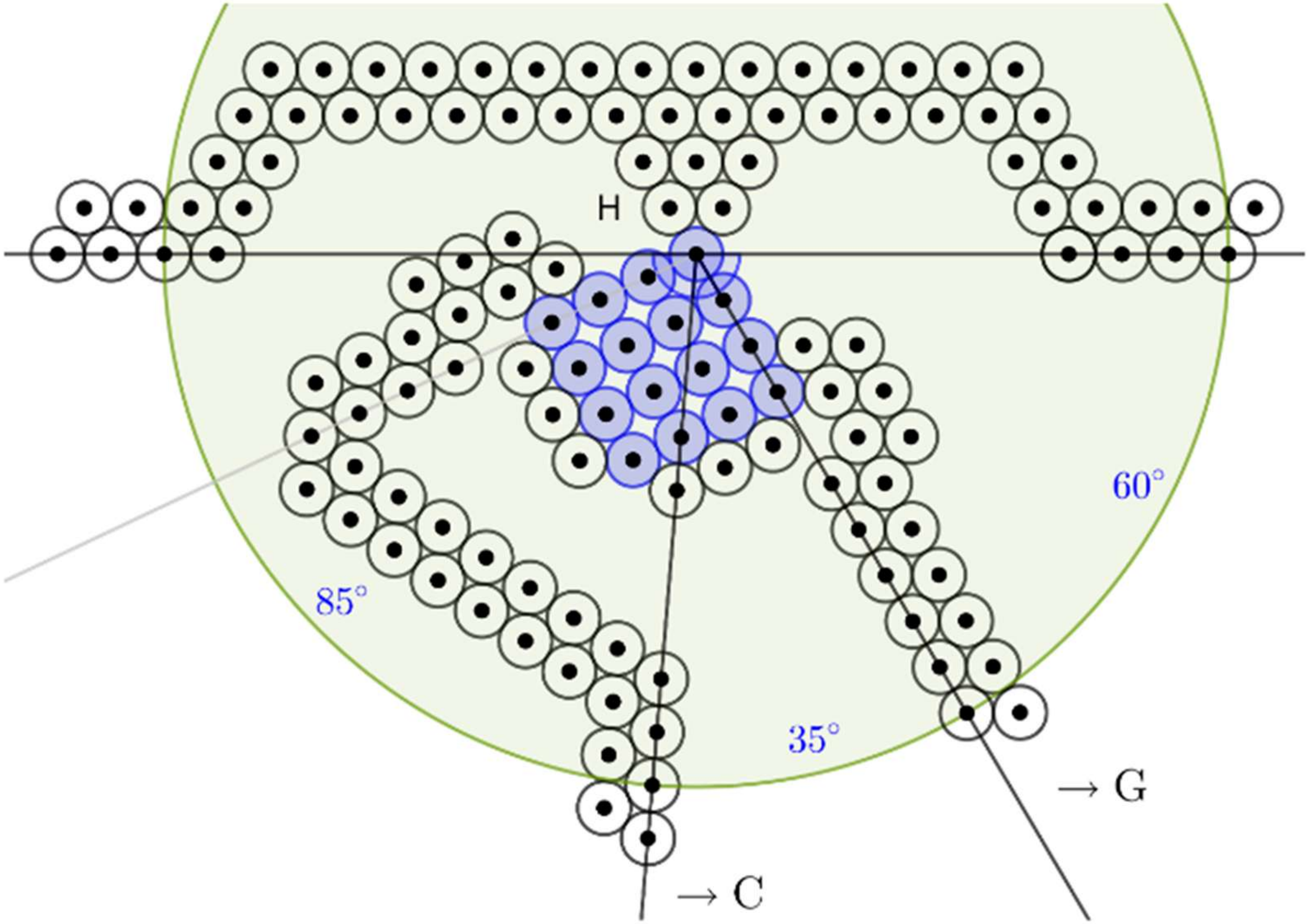


Flexible Angle Vertices II.V



Christoph Scheiner, *Pantographice*, 1636
https://catalog.hathitrust.org/Record/100877903

Flexible Angle Vertices III



qed

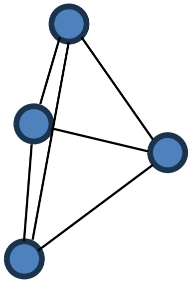
Rigidity

Theorem

Testing rigidity of a penny graph configuration is $\forall\mathbb{R}$-complete.

Marble Graphs

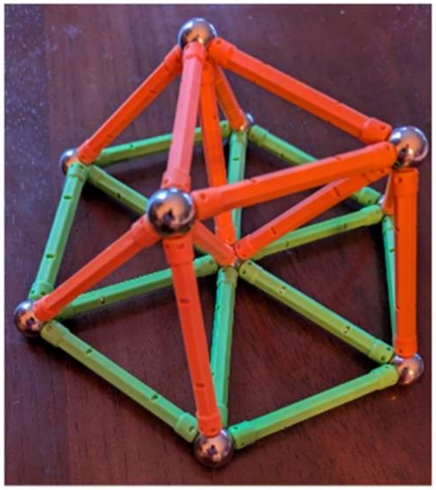
$K_4 \longrightarrow$

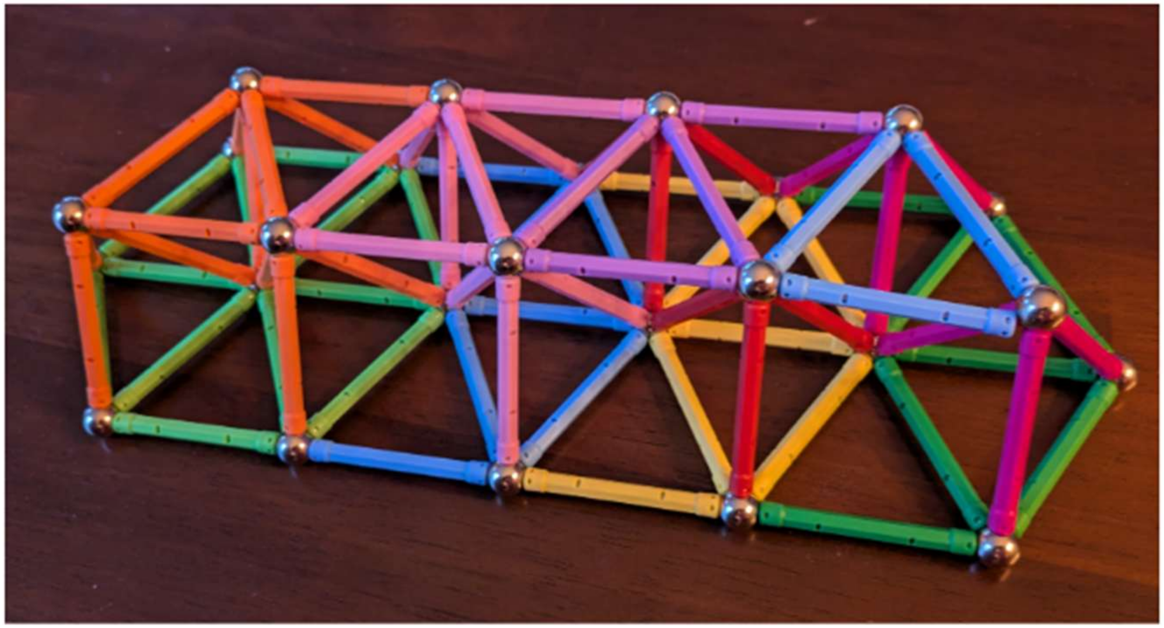


Theorem (Lubiw, Schaefer, 2025)

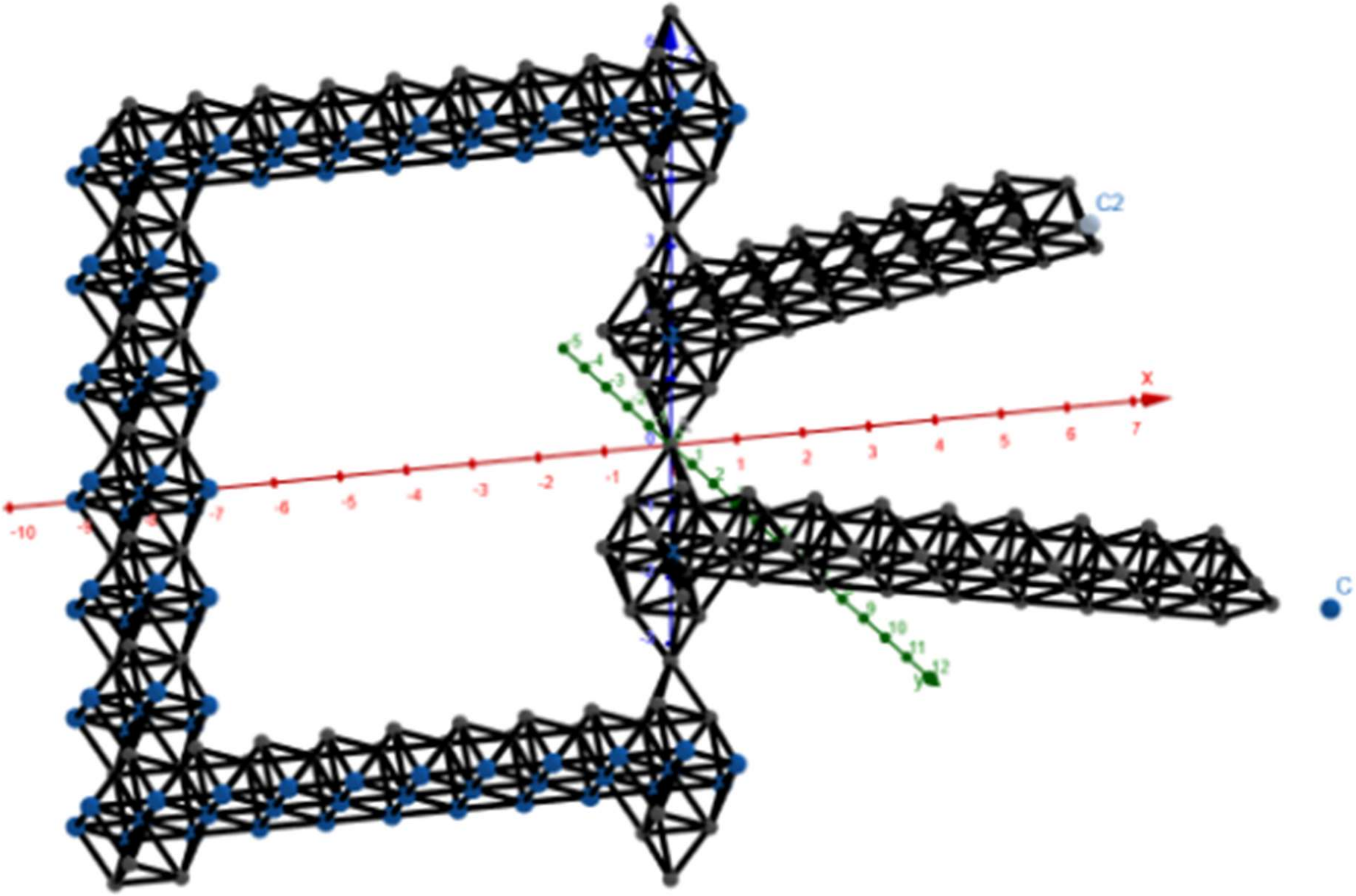
Recognizing marble graphs is $\exists\mathbb{R}$-complete.

Pennies → Marble Gadgets I





Pennies → Marble Gadgets II



Contacts in Higher Dimensions

dim	unit ball contact graphs	ball contact graphs
2	∃ℝ-complete ↘	P
3	∃ℝ-complete ↘	∃ℝ-complete
4	NP-hard (Hliněný, 1997) ↘	∃ℝ-complete
5		NP-hard
8	NP-hard (Hliněný, Kratochvíl, 2001) ↘	
9		NP-hard
24	NP-hard (Hliněný, Kratochvíl, 2001) ↘	
25		NP-hard

↘

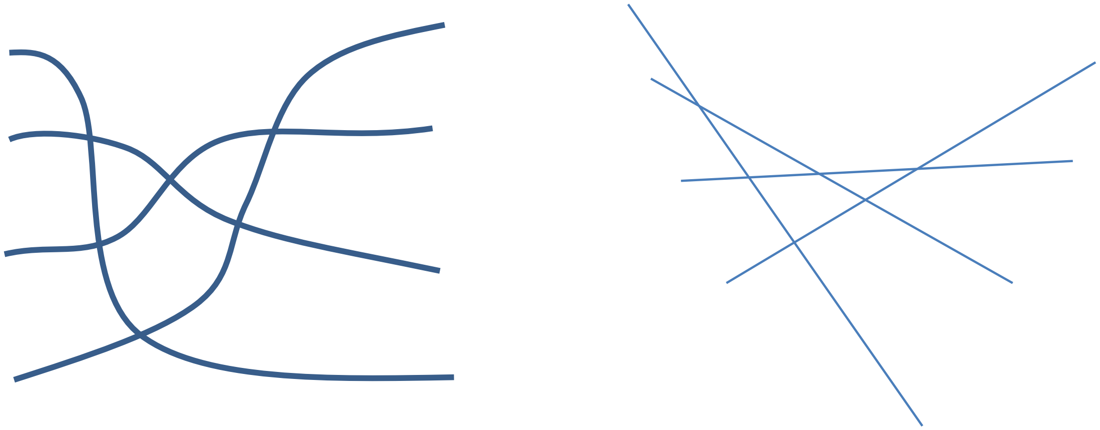
Proposition (Kirkpatrick, Rote)
G unit ball contact graph in $\mathbb{R}^d$
iff
$G + K_2$ ball contact graph in $\mathbb{R}^{d+1}$

The Dark Side

$$\exists x, y, z \colon x^2 = x \land y^2 = y \land x < y \land z^2 = y + y$$

$$\begin{aligned} \exists x_1, \ldots, x_n \colon \quad & 0 < \ x_1 < \frac{1}{2} \\ & \land 0 < x_2 < x_1^2 \land \\ & \ldots \\ & \land 0 < x_n < x_{n-1}^2 \end{aligned}$$

Universality



Theorem (Mnëv, 1988, also Shor, 1991)

Every semialgebraic set is “stably equivalent”
to the realization space of an arrangement.

or Matoušek, 2014 !

Precision and Grids

Theorem

Penny graphs of order n which are trees can be realized on a grid of size $2^{2^{n^k}} \times 2^{2^{n^k}}$ for constant $k > 0$.

Precision and Grids

Theorem

Penny graphs of order n which are trees can be realized on a grid of size $2^{2^{n^k}} \times 2^{2^{n^k}}$ for constant $k > 0$.

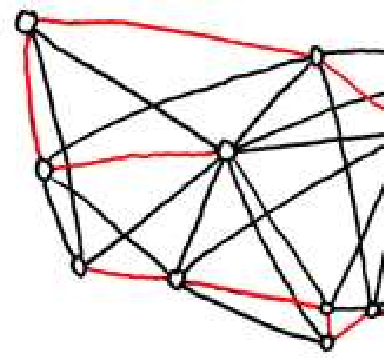
True for bipartite graphs?

Thank You

WHAT ABOUT EUCLIDEAN TSP, MINIMUM WEIGHT TRIANGULATION, ETC?

Euclidean Travelling Salesman

Theorem (Garey, Graham, Johnson, 1976)



is NP-hard.

NP-hard ?

The Sum of Square Roots

$$\sqrt{3}+\sqrt{5}\overset{?}{<}\sqrt{7}$$

The Sum of Square Roots

$$\sqrt{3}+\sqrt{5}\overset{?}{<}\sqrt{7}$$

$$\sqrt{3}+\sqrt{5}+\sqrt{8}+\sqrt{12}+\sqrt{13}+\sqrt{14}\overset{?}{<}\sqrt{6}+\sqrt{10}+\sqrt{15}+\sqrt{16}+\sqrt{17}$$

The Sum of Square Roots

$$\sqrt{3}+\sqrt{5} \overset{?}{<} \sqrt{7}$$

$$\sqrt{3}+\sqrt{5}+\sqrt{8}+\sqrt{12}+\sqrt{13}+\sqrt{14} \overset{?}{<} \sqrt{6}+\sqrt{10}+\sqrt{15}+\sqrt{16}+\sqrt{17}$$

17.60785618719 17.60785637477

The Sum of Square Roots

$$\sqrt{3}+\sqrt{5} \overset{?}{<} \sqrt{7}$$

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17.60785618719 17.60785637477

The Sum of Square Roots

$$\sqrt{3}+\sqrt{5}+\sqrt{8}+\sqrt{12}+\sqrt{13}+\sqrt{14}<\sqrt{6}+\sqrt{10}+\sqrt{15}+\sqrt{16}+\sqrt{17}$$

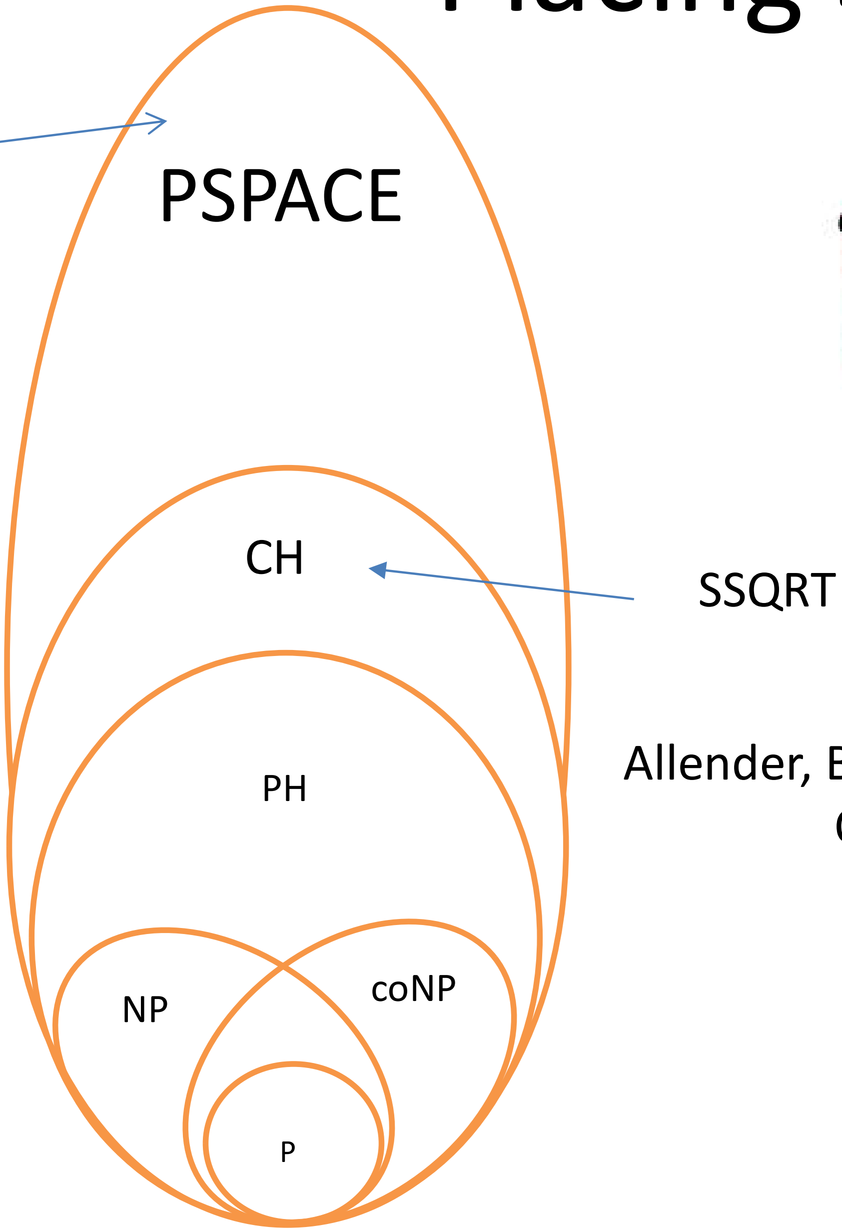
$$r(n,k) := \min \left| \sum_{i=1}^{k} \sqrt{a_i} - \sum_{i=1}^{k} \sqrt{b_i} \right|$$

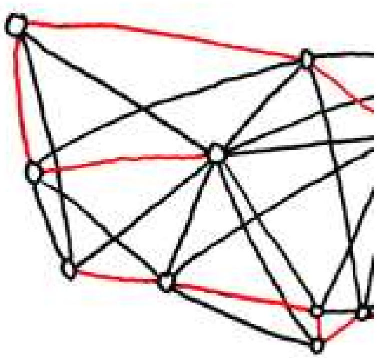
nonzero, a_i, $b_i \leq n$

$$\Omega(2k \lg n) \leq -\lg r(n,k) \leq O(2^{2k} \lg n)$$

http://maven.smith.edu/~orourke/TOPP/P33.html

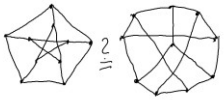
Placing SSQRT



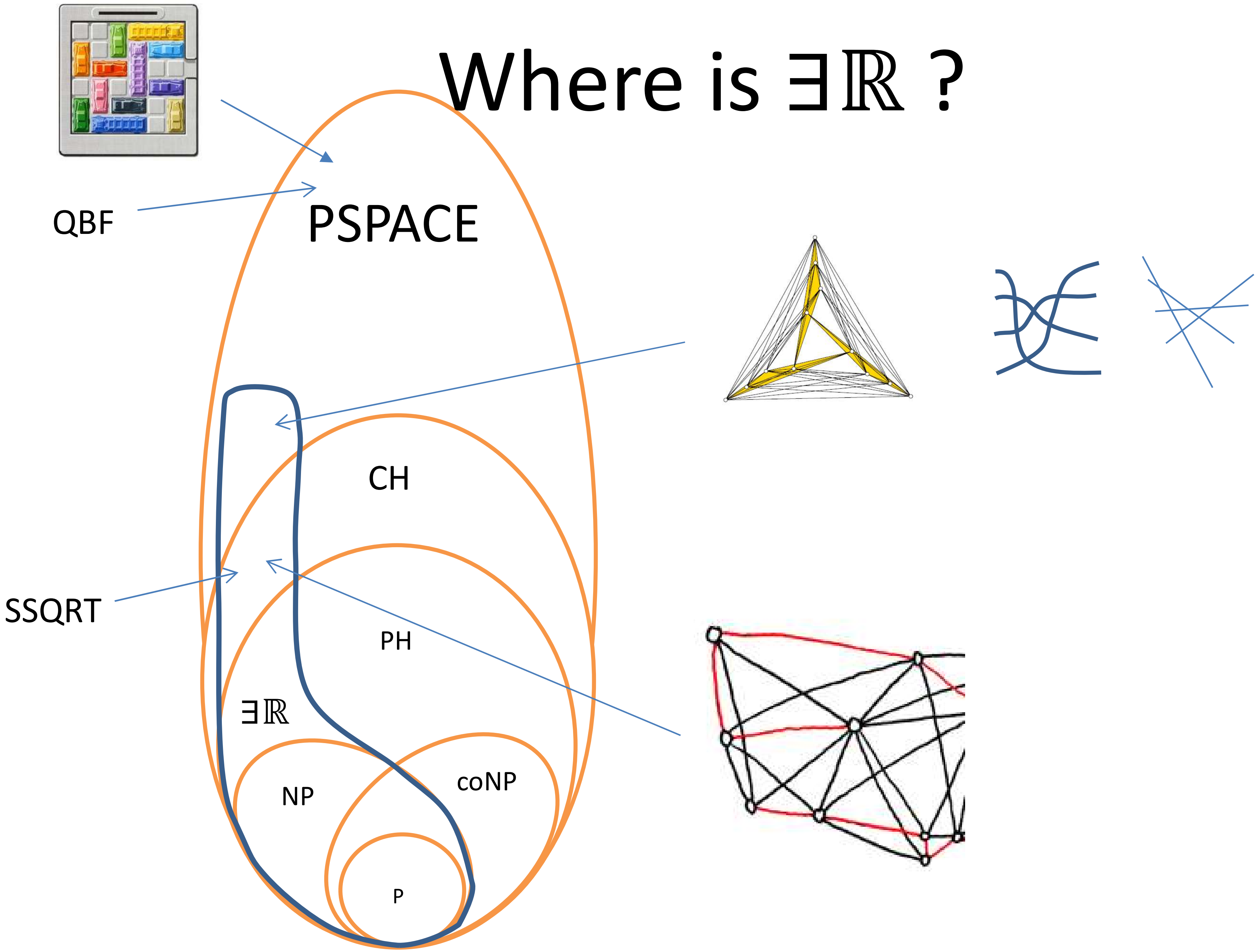


in NP^{SSQRT}

Allender, Burgisser, Kjeldgaard-Pedersen, Miltersen
On the Complexity of Numerical Analysis, 2008

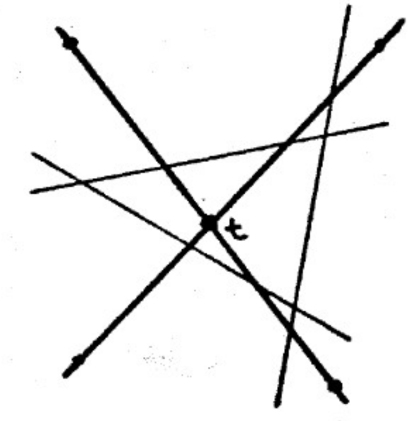


Where is ∃ℝ ?



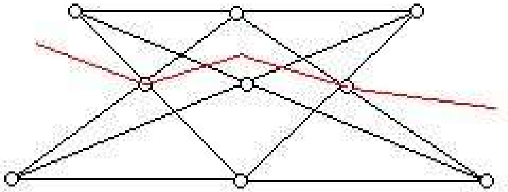
A Subtlety

$\exists_{<}\mathbb{R}$



- rcr
- segment intersection

$\exists_{=}\mathbb{R}$



- Nash equilibrium
- Steinitz problem
- Euclidean dimension

Theorem (S, Štefankovič, 2017)

$$\exists_{<}\mathbb{R} = \exists_{=}\mathbb{R}$$